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EQUILIBRIA IN BANACH LATTICES WITHOUT ORDERED PREFERENCES

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This paper establishes a very general result on the existence of competitive equilibria for exchange economies (with a finite number of agents) with an infinite-dimensional commodity space. The commodity spaces treated are Banach lattices, but no interiority assumptions on the positive cone are made; thus, the commodity spaces covered by this result include most of the spaces considered in economic analysis. Moreover, this result applies to preferences which may not be monotone, complete, or transitive; preferences may even be interdependent. Since preferences need not be monotone, the result allows for non-positive prices, and an exact equilibrium is obtained, rather than a free-disposal equilibrium.

1. Introduction

Infinite-dimensional commodity spaces have become well-established in the literature since their introduction by Debreu (1954), Peleg and Yaari (1970) and Bewley (1972, 1973). Infinite-dimensional commodity spaces arise naturally when we consider economic activity over an infinite time horizon, or with uncertainty about the (possibly infinite number of) states of the world, or in a setting where an infinite variety of commodity characteristics are possible. Many different infinite-dimensional spaces arise naturally. For example, Bewley (1972) uses the space l_∞ of bounded real sequences to model

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the allocation of resources over an infinite time horizon,[1] and the Lebesgue space L_∞ of bounded measurable functions on a measure space to model uncertainty. Duffie and Huang (1985) use the space L_2 of square-integrable functions on a measure space to model the trading of long-lived securities over time. Finally, Mas-Colell (1975) and Jones (1983) use the space $M(\Omega)$ of measures on a compact metric space to model differentiated commodities.

This paper establishes a very general result on the existence of competitive equilibria for exchange economies (with a finite number of agents) with an infinite-dimensional commodity space. The commodity spaces we treat are Banach lattices, and include all the sequence spaces l_p $(1 \leqq p \leqq \infty)$, all the Lebesgue spaces L_p $(1 \leqq p \leqq \infty)$ and the space $M(\Omega)$ of measures. Thus we allow for commodity spaces which are general enough to include most of the spaces used in economic analysis. Moreover, we allow for preferences which may not be monotone, transitive or complete; preferences may even be interdependent. Since preferences need not be monotone, we allow for prices which need not be positive, and obtain an exact equilibrium rather than a free-disposal equilibrium.

The central assumption we make is that preferences satisfy the non-transitive version of a condition used by Mas-Colell (1983), which he called 'uniform properness' [and which is, in turn, related to a condition used by Chichilnisky and Kalman (1980)]. Informally, preferences satisfy this condition if there is one commodity bundle which is a uniformly good substitute for any other commodity bundle (in appropriate quantities). This assumption is quite weak; it is automatically satisfied, for example, whenever preferences are monotone and the positive cone has a non-empty interior. (This includes all finite-dimensional spaces and the infinite-dimensional spaces l_∞ and L_∞.) It also admits many natural economic interpretations; for example, in infinite time horizon models it corresponds to the assumption that agents do not over emphasize the future.

Our work is closely related to the work of Mas-Colell (1983), in the sense that our crucial assumption is analogous to his. However, Mas-Collell's argument [which is related to an idea of Magill (1981) and Negishi (1960)] depends crucially on completeness and transitivity of preferences. On the other hand, the arguments of Bewley (1972), which have been generalized by Florenzano (1983), Toussaint (1984), Khan (1984) and others, depend crucially on monotonicity (or free disposability) and on the assumption that the positive cone of the commodity space has a non-empty interior. (Of the commodity spaces mentioned previously, only l_∞ and L_∞ enjoy this pro-

[1]The space l_1 of summable sequences can also be used for such a model. Our choice between l_∞ and l_1 should be based on the sort of resources we have in mind. If we are considering a renewable resource (such as food) we should use l_∞, since the finiteness of the earth places an upper bound on the amount available in any time period. If we are considering a non-renewable resource (such as oil), it seems more appropriate to use l_1, since not only the amount available in each period, but also the sum total available throughout time is (presumably) bounded.

perty.) Since we do not assume completeness or transitivity or monotonicity of preferences, and make no interiority assumptions on the positive cone of our commodity space, our arguments are of necessity quite different. At the heart of our proof is a price estimate which says that, at equilibrium, commodities which are very desirable cannot be cheap.

The lattice framework of our paper is superficially similar to that used by Aliprantis and Brown (1983) [see also Bojan (1974) and Yannelis (1985)]. However, these authors take, as the primitive notion, the aggregate exceess demand function (or correspondence). Since our primitive notion is that of agents' preferences, the two approaches are not comparable. It is perhaps appropriate to point out however, as Aliprantis and Brown (1983, p. 196) point out, that if the interior of the positive cone of the commodity space is empty, the equilibrium price they obtain may be zero, which is not economically meaningful. By contrast, our equilibrium prices are never zero.

The paper is organized as follows: the model is described in section 2 (in a standard way); we also give some motivation for our use of Banach lattices as commodity spaces. Section 3 discusses the economic and mathematical meaning of our assumptions on preferences.

The Main Existence Theorem is presented in section 4. We formulate this result in a very general context so that more concrete results flow naturally and easily from it. Since the proof of this result is long, section 4 includes an overview of the proof, together with a detailed discussion of the failure of more traditional approaches. We think both this overview and this discussion are important for understanding the proof.

The proof of the Main Existence Theorem is spread out over 3 sections. Section 5 contains the key economic lemma, dealing with prices. Section 6 contains the key mathematical lemma, dealing with finite-dimensional vector sublattices. Section 7 completes the argument.

We collect a few concluding remarks in section 8. Finally, the appendix reviews some standard material about Banach spaces and Banach lattices.

2. Economies in a Banach lattice

We formalize the notion of an economy in the usual way. Let L be a Banach lattice.[2] By an *exchange economy with N agents and commodity space L* (or simply an *economy in L*) we mean a set $\mathcal{E}=\{(X_i, P_i, e_i): i=1,2,\dots,N\}$ of triples where

(a) X_i (the *consumption set of the ith agent*) is a non-empty subset of L,
(b) P_i (the *preference relation of the ith agent*) is a correspondence $P_i\colon \prod_{j=1}^{N} X_j \to 2^{X_i}$ (2^{X_i} is the set of all subsets of X_i),
(c) e_i (the *initial endowment of the ith agent*) is a vector in X_i.

[2]For background information about Banach lattices, see the appendix.

We frequently refer to a vector $(x_1,\dots,x_N)\in\prod X_j$ as an *allocation.* The interpretation of preferences which we have in mind is that $y_i\in P_i(x_1,\dots,x_N)$ means that agent i strictly prefers y_i to x_i if the (given) components of other agents are fixed; this is the usual way to allow for interdependent preferences. Notice that preferences need not be transitive or complete or convex. However, in all our results we shall asume that $x_i\notin \operatorname{con} P_i(x_1,\dots,x_N)$ for all $(x_1,\dots,x_N)\in\prod X_j$ ($\operatorname{con} A$ always denotes the convex hull of the set A); in particular, $x_i\notin P_i(x_1,\dots,x_N)$ so P_i is *irreflexive.*

The graph of the correspondence P_i is a subset of $\prod_{j=1}^N X_j\times X_i$. If τ is a topology on L, we shall say that P_i is (τ, norm)-*continuous* if the graph of P_i is an open set of the product $\prod_{j=1}^N X_j\times X_i$, where we endow each of the first N factors with the topology τ and the last factor X_i with the norm topology (product spaces will always be given the product topology). This is equivalent to saying that if $y_i\in P_i(x_1,\dots,x_N)$ then there are relatively τ-open neighborhoods U_j of x_j in X_j and relatively norm-open neighborhood V_i of y_i in X_i such that $\bar{y}_i\in P_i(\bar{x}_1,\dots,\bar{x}_N)$ whenever $\bar{y}_i\in V_i$ and $\bar{x}_j\in U_j$ for each $j=1,2,\dots,N$. Mixed continuity is common in infinite-dimensional settings; see Bewley (1972) for example. The topology τ we shall use will be different in different settings; we refer to section 4 for further discussion.

A *price* is a continuous linear functional π on L (i.e., $\pi\in L'$). By an *equilibrium* for the economy $\mathcal{E}$ we mean an $(N+1)$-tuple $(x_1,\dots,x_N;\pi)$ where $x_i\in X_i$ for each i and π is a non-zero price, such that

(i) $\sum_{i=1}^N x_i=\sum_{i=1}^N e_i$,
(ii) $\pi(x_i)=\pi(e_i)$ for each i,
(iii) if $y_i\in P_i(x_1,\dots,x_N)$ then $\pi(y_i)>\pi(e_i)$ (for each i).

(Notice that we do not require prices to be positive and that we treat *exact* equilibria rather than free disposal equilibria.) A *quasi-equilibrium*[3] is an $(N+1)$-tuple $(x_1,\dots,x_N;\pi)$ where $x_i\in X$ for each i, and π is a non-zero price, such that (i), (ii) above and the following hold:

(iii′) if $y_i\in P_i(x_1,\dots,x_N)$ then $\pi(y_i)\geqq\pi(e_i)$ (for each i).

We have restricted our attention to continuous prices because that seems economically natural. However, in Yannelis–Zame (1984) we show that, in the context we consider in this paper, discountinuous prices can safely be ignored. That is, allocations which can be supported in equilibrium by discontinuous prices can also be supported in equilibrium by continuous prices.

[3]Strictly speaking, this defines a compensated equilibrium, rather than a quasi-equilibrium. However, in the presence of our other assumptions, these two notions are equivalent.

We shall say that the economy $\mathcal{E}$ is *irreducible* if: whenever I and J are non-empty sets of agents with $I \cap J = \varnothing$ and $I \cup J = \{1,\ldots,N\}$, and $(x_1,\ldots,x_N)$ is an allocation such that $\sum_{i=1}^N x_i = \sum_{i=1}^N e_i$, then there is an agent $m \in I$, an agent $n \in J$ and a vector $\zeta \in L$ with $0 \leqq \zeta \leqq e_n$ and $x_m + \zeta \in P_m(x_1,\ldots,x_N)$. [See McKenzie (1959).]

We will frequently refer to a vector $x \in L^+$ as a *commodity bundle*. We should caution the reader that, in our abstract framework, *there are no pure commodities.*

Finally, we make one comment about our use of Banach lattices as commodity spaces. It might seem more natural (and less restrictive) to use ordered Banach spaces, rather than lattices, as commodity spaces. However, many economic ideas lose their natural meanings if the lattice structure is missing. Suppose for example that the economy has two agents with initial endowments e_1 and e_2, and we consider the meaning of the statement 'the (positive) commodity bundle b is part of the aggregate initial endowment'. Presumably this should mean $0 \leqq b \leqq e_1 + e_2$. On the other hand, we should also like it to have the meaning that the whole bundle b is the sum of two parts, one of which is owned by each agent. In other words, there should exist bundles b_1, b_2 with $0 \leqq b_1 \leqq e_1, 0 \leqq b_2 \leqq e_2$ and $b = b_1 + b_2$. Unfortunately, if the commodity space *is not a lattice*, these two statements are *not equivalent*. On the other hand, if the commodity space *is a lattice*, these two statements *are equivalent* (this is just the Riesz Decomposition Property).

3. Preferences

The purpose of this section is to discuss in detail our key assumptions on preferences and their meaning. Throughout the remainder of this section, we let

$$\mathcal{E} = \{(X_1, P_i, e_i) : i = 1, 2, \ldots, N\}$$

be an economy in the Banach lattice L. We will assume that each of the consumption sets X_i coincides with the positive cone L^+ of L (in section 8 we discuss ways in which this assumption can be weakened) and that $x_i \notin \operatorname{con} P_i(x_1,\ldots,x_N)$ for each agent i and each $(x_1,\ldots,x_N) \in \prod_{j=1}^N X_j$.

Fix an agent i, a vector $v \in L^+$ and an allocation $x = (x_1,\ldots,x_N) \in \prod X_j$. Let $\Gamma_v^i(x)$ denote the set of non-negative real numbers μ such that:

$$x_i + tv - \sigma \in P_i(x_1,\ldots,x_N) \quad \text{whenever} \quad 0 < t \leqq 1,$$

$$\sigma \leqq x_i + tv \quad \text{and} \quad \|\sigma\| < t\mu.$$

(The restriction $\sigma \leqq x_i + tv$ guarantees that $x_i + tv - \sigma \geqq 0$ so $x_i + tv - \sigma \in X_i = L^+$.) It is easily checked that $\Gamma^i_v(x)$ is a closed interval containing 0 and bounded above by $\|v\|$.

Definition The *marginal rate of desirability of* v (*for agent* i) *at* x is

$$\mu_i(v, x) = \max\{\mu : \mu \in \Gamma^i_v(x)\}.$$

If A is a subset of $\prod X_j$, we say the vector v is *extremely desirable* (*for agent* i) *on the set* A *if*

$$\inf\{\mu_i(v, x) : x \in A\} > 0.$$

Finally, v is *extremely desirable* (*for agent* i) if it is extremely desirable on $(L^+)^N$.

Informally, v is extremely desirable if agent i would prefer to trade any bundle σ for an additional increment of the bundle v, provided that the size of σ (measured by $\|\sigma\|$) is sufficiently small compared to the increment of v (measured by t). We stress that – even in those contexts where it makes sense to speak of 'pure commodities' – the vector v need not be a pure commodity; but rather a commodity bundle. Evidently, extreme desirability is a kind of bound on the relative marginal rates of substitution, where we compare v to all other bundles.

As will become clear in the following sections, existence of extremely desirable commodities (for each agent) is precisely the additional assumption we need to obtain existence of equilibria, so its seems valuable to discuss the meaning of this assumption in some detail.

Let us observe first of all that if the preference relation P_i is strictly monotone [in the sense that $x_i + y \in P_i(x_1, \ldots, x_N)$ whenever y is strictly positive] and the positive cone L^+ has a non-empty interior, then extremely desirable commodities exist automatically. Indeed, let v be any vector in the interior of L^+ and choose a positive number μ such that the ball $B = \{w \in L; \|v - w\| < \mu\}$ is contained in the interior of L^+. Now, if $\|\sigma\| < t\mu$ then $\|t^{-1}\sigma\| < \mu$ so $tv - \sigma = t(v - t^{-1}\sigma)$ belongs to the interior of L^+. (for $t > 0$). Strict monotonicity now implies that $x_i + tv - \sigma \in P_i(x_1, \ldots, x_N)$. (Informally, $x_i + tv - \sigma$ is better than x_i because it is strictly bigger.) Since the positive cone L^+ has a non-empty interior for every finite-dimensional space L, and for l_∞ and L_∞, requiring existence of extremely desirable commodities imposes no additional restriction in these cases.

Extreme desirability may be given a very natural geometric interpretation. Fix a vector v in L^+ and a positive number μ, and let C be the open cone

$$C = \{tv - \sigma : 0 < t \leqq 1, \sigma \in L, \|\sigma\| < t\mu\}.$$

The vector v is extremely desirable (for agent i) with marginal rate of desirability at least μ, if for each x in $(L^+)^N$; it is the case that $y \in P_i(x)$ whenever y belongs to $(C+x_i) \cap L^+$. In other words, extreme desirability means that the portion of the forward cone $C+x_i$ which belongs to the consumption set of consumer i is contained in the set of vectors preferred to x_i (keeping other components of x fixed).

By way of comparison, Mas-Colell (1983) says that the (transitive, complete, convex) preference relation $\succsim_i$ is *uniformly proper* if there is a vector v in L^+ and a positive real number μ such that $(x_i - tv + \sigma) \not\succsim_i x$ whenever $x_i \in L^+, t > 0$, $\sigma \in L$ and $\|\sigma\| < t\mu$. Since it is automatically the case that $(x_i - tv + \sigma) \not\succsim_i x$ if $(x_i - tv + \sigma) \notin L^+$, this is equivalent to saying that $[(-C+x_i) \cap L^+] \cap \{y_i: y_i \succsim_i x_i\} = \emptyset$. Thus uniform properness means that the portion of the backward cone $-C+x_i$ which belongs to the consumption set of consumer i is disjoint from the set of vectors preferred to x_i.

It should be evident, then, that the existence of extremely desirable commodities is simply the non-transitive analog of uniform properness. In fact, it may be shown [see Yannelis and Zame (1984) for the easy argument] that – for transitive, complete, convex, non-interdependent preferences – the two conditions are equivalent. All of Mas-Colell's comments on the meaning of uniform properness thus apply to extreme desirability; we shall not repeat them here. Nor shall we repeat Mas-Colell's example which shows that, without uniform properness (i.e., in the absense of extremely desirable commodities), an economy may fail to have an equilibrium. It does, however, seem natural to give one example to illustrate the economic meaning of extremely desirable commodities.

Example 3.1. Let $(\Omega, \mathscr{R}, m)$ be a measure space with m a positive measure such that $m(\Omega) = 1$. We wish to think of Ω as representing the set of possible states of the world, so that $m(E)$ is the probability that the true state of the world is one of the states in the set E, with $E \in \mathscr{R}$. We interpret a function $f \in L_1^+$ as representing the allocation of a single resource over all possible states of the world so that $\|f\| = \int_\Omega f \, dm$ is the consumer's expected allocation of this one resource. Take v to be the function which is identically equal to 1, so that v represents a guarantee of one unit of the resource no matter what the true state of the world is. If $\|\sigma\| = \int_\Omega |\sigma| \, dm$ is small in comparison with t, then $x + tv - \sigma$ represents a guaranteed gain of t units of the resource in every state of the world, and a loss of an amount which, although perhaps large in some states of the world, is expected to be small (in comparison with t). To say that the bundle v is extremely desirable (with some marginal rate of desirability) is thus to place a bound on the degree to which the consumer is 'risk-preferring'.

One further comment of a mathematical nature. Notice that the marginal rate of desirability $\mu_i(v,x)$ depends both on the bundle v and on the allocation x. As a function of x, $\mu_i(v,x)$ need not be continuous, but as a function of v we have the following easy estimate which we shall need later:

Lemma 3.2. Let v and w belong to L^+ and let x belong to $\prod_{j=1}^N X_j$. Then:

$$\mu_i(w,x) \geqq \mu_i(v,x) - \|v-w\|.$$

Proof. Let t be a real number with $0<t\leqq 1$ and let σ be an element of L with $\sigma \leqq x_i + tw$ and $\|\sigma\| \leqq t\mu_i(v,x) - t\|v-w\|$; we wish to show that $x_i + tw - \sigma \in P_i(x)$. Write $x_i + tw - \sigma = x_i + tv - (\sigma + tv - tw)$ and observe that $\sigma + tv - tw \leqq x_i + tv$ (since $\sigma \leqq x_i + tw$ and $tv \geqq 0$) and that

$$\begin{aligned}\|\sigma + tv - tw\| &\leqq \|\sigma\| + t\|v-w\| \\ &\leqq t\mu_i(v,x) - t\|v-w\| + t\|v-w\|,\end{aligned}$$

so that $\|\sigma + tv - tw\| < t\mu_i(v,w)$. Of course this means that $x_i + tw - \sigma = x_i + tv - (\sigma + tv - tw)$ belongs to $P_i(x)$ as desired. □

4. The Main Existence Theorem

In this section we formulate a very general existence result from which we can easily derive concrete applications. We begin with a definition.

Definition. A Hausdorff topology τ on the Banach lattice L will be called *compatible* if

(a) τ is weaker than the norm topology of L,
(b) τ is a vector space topology (i.e., the vector space operations on L are continuous in the topology τ),
(c) all order intervals $[0,z]$ in L are τ-compact.

Note that we do *not* assume that the lattice operations in L are continuous in the topology τ. In concrete applications, the topology τ will vary according to the underlying Banach lattice L; it may be the norm topology itself, or the weak topology, or the weak-star or Mackey topology (if L is a dual space).

Our basic existence result is the following:

Main Existence Theorem. Let $\mathcal{E} = \{(X_i, P_i, e_i)$: $i=1,2,\ldots,N\}$ be an economy in the Banach lattice L, and let τ be a compatible topology on L. Assume that:

(1) $X_i = L^+$ for each agent i,

(2) *the aggregate initial endowment* $e=\sum_{i=1}^{N} e_i$ *is strictly positive,*
(3) $x_i \notin \operatorname{con} P_i(x_1,\ldots,x_N)$, *for each agent i and each point* $(x_1,\ldots,x_N)$ *in* $(L^+)^N$.
(4) *for each agent i, there is a commodity* $v_i \in L^+$ *which is extremely desirable for agent i on the set*

$$\mathscr{A}=\left\{x=(x_1,\ldots,x_N): x\in(L^+)^N,\ \sum_{i=1}^{N} x_i \leqq e\right\}$$

of feasible allocations,
(5) *each of the preference relations* P_i *is* (τ, norm)*-continuous.*

Then $\mathcal{E}$ *has a quasi-equilibrium* $(\bar{x}_1,\ldots,\bar{x}_N;\bar{\pi})$ *with the price* $\bar{\pi}$ *belonging to* L'. *If* $\mathcal{E}$ *is irreducible, then every quasi-equilibrium is an equilibrium.*[4,5]

In concrete settings, the choice of compatible topology will be dicated by the underlying commodity space L. For instance, if $L=l_p(1\leqq p<\infty)$ [the space of real sequences (a_n) such that $\|(a_n)\|_p=(\sum|a_n|^p)^{1/p}<\infty$], then the norm topology itself is compatible [since order intervals in l_p are norm compact – see Yannelis and Zame (1984) for a proof]. If $L=L_p(1\leqq p<\infty)$ (the space of equivalence classes of pth power integrable functions on a measure space) then the weak topology is compatible [since order intervals are weakly compact – see Schaefer (1974, pp. 90–92, 119)]. If $L=l_\infty$ or L_∞, then the weak-star topology is compatible (since order intervals are weak-star closed and bounded, hence weak-star compact by Alaoglu's Theorem). With the appropriate choice of compatible topology, the Main Existence Theorem simply applies verbatim for economies in any of these commodity spaces.

If the commodity space is $M(\Omega)$ (the space of regular Borel measures on the compact space Ω), the weak-star topology is again compatible. However, the Main Existence Theorem may not be applicable since it requires that the aggregate initial endowments be strictly positive, and $M(\Omega)$ need not have any strictly positive elements. However, it is possible to adapt our result to cover this case – see Remark 4 of section 8 for details.

The formal proof of the Main Existence Theorem is long and involved; we defer it to the following sections. At this point, however, it is appropriate to give an overview of the proof.

It is helpful to recall the argument used by Bewley (1972) (and generalized by others) for the case $L=L_\infty$. In sketch, the strategy of Bewley's proof is to consider the restriction $\mathcal{E}^F$ of the economy $\mathcal{E}$ to finite-dimensional subspaces

[4]We have required extreme desirability on the set of feasible allocations, rather than all of $(L^+)^N$, since that is all we shall need, and it is a bit easier to verify in practice. See Yannelis and Zame (1984) for example.

[5]Note that the price π is norm continuous but need not be continuous in the topology τ. Indeed, τ need not even admit any non-zero continuous linear functionals.

F of $L=L_\infty$ which contain the initial endowments. Standard results imply that each of the economies $\mathcal{E}^F$ has an equilibrium $(x_1^F,\dots,x_N^F;p^F)$ with $p^F\in F'$. Since Bewley assumes that preferences are monotone, it is necessarily the case that $p^F\geqq 0$ and there is no loss of generality in assuming that $\|p^F\|=1$. Since e is strictly positive, it is in fact an interior point of the positive cone of $L=L_\infty$. The Krein–Rutman Theorem then allows us to extend p^F to an element π^F of $L'=L'_\infty$ with $\pi^F\geqq 0$ and $\|\pi^F\|=1$. The net of equilibria $(x_1^F,\dots,x_N^F;\pi^F)$ has a subnet which converges (in the respective weak-star topologies) to $(\bar{x}_1,\dots,\bar{x}_N;\bar{\pi})$. Since the functionals π^F are positive and have norm 1, non-emptiness of the interior of the positive cone implies that the (weak-star) limit functional $\bar{\pi}$ is also positive and also has norm 1. In particular, $\bar{\pi}$ is not the zero functional. It now follows that $(\bar{x}_1,\dots,\bar{x}_N;\bar{\pi})$ is an equilibrium for $\mathcal{E}$.

This argument depends crucially *both* on the assumption that preferences are monotone *and* on the assumption that the positive cone of the Banach lattice L has a non-empty interior; it will not work if *either* of these assumptions is dropped. The problem is that we must be sure that the limit price $\bar{\pi}$ is not identically zero. If preferences are not monotone, we cannot be sure that the prices p^F (and hence their extensions π^F) are positive. Since the functionals π^F only converge to $\bar{\pi}$ in the weak-star topology, however, there is then no reason to suppose that $\bar{\pi}$ is not identically zero. (This can happen in the dual of *any* infinite-dimensional Banach space, including L_∞.) On the other hand if the positive cone of L has an empty interior, then the limit functional $\bar{\pi}$ may again be zero – even if all the functionals π^F are positive and have norm 1.

The purpose of this discussion is to point out that the crucial issue is to guarantee that the limit functional $\bar{\pi}$ is not identically zero. The central idea of our proof is to consider, not finite-dimensional subspaces of L, but rather finite-dimensional vector sublattices. For vector sublattices, we can use the extremely desirable commodities v_i to obtain an estimate (which we call the Price Lemma, and isolate in section 5) which will, in the limit, guarantee that $\bar{\pi}$ is not identically zero. However, this approach creates a multitude of its own difficulties. The first difficulty is that, in general, a Banach lattice need not have 'enough' finite-dimensional vector sublattices; we take care of this in section 6 by showing that the existence of a compatible topology implies the existence of 'many' finite-dimensional vector sublattices. The second difficulty is that, even with an abundance of finite-dimensional vector sublattices, we cannot be sure of finding *any* finite dimensional vector sublattices which contain the initial endowments; we take care of this by constructing economies in the finite-dimensional vector sublattices which are approximations of the original economy, rather than restrictions of it. The third difficulty is that the family of finite-dimensional vector sublattices is not directed by inclusion; we take care of this by directing them by 'approximate inclusion' instead. The final difficulty lies in showing that the limiting allocation is an equilibrium allocation, since it need not lie in any of the approximating economies; we take care of this by another approximation argument.

5. The Price Lemma

As we discussed in the previous section, the crucial issue in our argument is that the limiting price we construct must be different from zero. To achieve this, we shall make use of the following lemma, which formalizes a very natural economic intuition: at equilibrium, commodities which are very desirable cannot be cheap.

Price Lemma. *Let L be a Banach lattice, let $\mathcal{E}=\{(X_i, P_i, e_i): i=1,2,\dots,N]$ be an economy in L and let $(x_1,\dots,x_N;\pi)$ be a quasi-equilibrium*[6] *for $\mathcal{E}$ with $\|\pi\|=1$. Assume that*

(1) $X_i=L^+$ for each i,
(2) $e=\sum_{i=1}^N e_i$ is strictly positive,
(3) for each i, there is a commodity $v_i \in L^+$ such that the marginal rate of desirability $\mu_i(v_i,(x_1,\dots,x_N))$ is not zero.

Then

$$\sum_{i=1}^{N} \frac{\pi(v_i)}{\mu_i(v_i,(x_1,\dots,x_N))} \geqq 1.$$

Proof. If this is not so, we will show how to construct vectors y_i which are all preferred to the given allocation and have the property that, for at least one agent i, y_i is cheaper than x_i; this will violate the quasi-equilibrium conditions.

To this end, we write $\mu_i=\mu_i(v_i,(x_1,\dots,x_N))$, and suppose that $\sum(\pi(v_i)/\mu_i)<1$. Since $\|\pi\|=1$, there is a vector $w\in L$ such that $\|w\|<1$ and $\pi(w)>\sum(\pi(v_i)/\mu_i)$. Write $w=w^+-w^-$. Since $e=\sum e_i$ is strictly positive, the sequences $\{ne \wedge w^+\}_{n=1}^{\infty}$ $\{ne \wedge w^-\}_{n=1}^{\infty}$ and converge in norm to w^+ and w^- respectively. Since π is norm continuous, we can choose a positive integer k so large that $\pi((ke \wedge w^+)-(ke \wedge w^-))>\sum(\pi(v_i)/\mu_i)$. Write

$$z=(ke \wedge w^+)-(ke \wedge w^-).$$

[6]Notice that the lattice L need not admit a compatible topology and that the preferences need not enjoy any continuity properties. In this generality, quasi-equilibria need not exist (in which case the Price Lemma is certainly true). Of course, in our applications we will make additional assumptions about L and P_i, but the Price Lemma seems to be of interest in itself, so we choose to give a proof in this more general setting.

Then $\|z\| \leqq \|w\| < 1$, $\bar{\pi}(z) > \sum(\pi(v_i)/u_i)$, $0 \leqq z^+ = ke \wedge w^+ < ke$ and $0 \leqq z^- = ke \wedge w^- \leqq ke$.

Since $(x_1, \dots, x_N; \pi)$ is a quasi-equilibrium for $\mathcal{E}$ we have that $\sum x_i = \sum e_i = e$ and $x_i \geqq 0$ for each i. Hence $z^+ \leqq \sum kx_i$ and $z^- \leqq \sum kx_i$. We can use the Riesz Decomposition Property to find vectors $a_1, \dots, a_N$, $b_1, \dots, b_N$ in L^+ such that $0 \leqq a_i \leqq kx_i$ and $0 \leqq b_i \leqq kx_i$ for each i, $z^+ = \sum a_i$ and $z^- = \sum b_i$. Notice that $0 \leqq a_i \leqq z^+$ and $0 \leqq b_i \leqq z^-$ for each i, so that $\|a_i - b_i\| \leqq \|z^+ - z^-\| = \|z\| < 1$.

We now define the desired vectors y_i by setting

$$y_i = x_i + \frac{1}{k\mu_i} v_i - \frac{1}{k}(a_i - b_i).$$

We assert that $y_i \in P_i(x_1, \dots, x_N)$ for each i. This of course follows from the definition of μ_i as the marginal rate of desirability, provided we verify that $(1/k)(a_i - b_i) \leqq x_i + (1/k\mu_i)v_i$ and that $\|(1/k)(a_i - b_i)\| < (1/k\mu_i) \cdot \mu_i$. The first of these inequalities follows from the facts that $a_i \leqq kx_i$, $b_i \geqq 0$ and $v_i \geqq 0$ (for each i); the second follows from the fact that, for each i,

$$\left\|\frac{1}{k}(a_i - b_i)\right\| = \frac{1}{k}\|a_i - b_i\| \leqq \frac{1}{k}\|z\| < \frac{1}{k} = \frac{1}{k\mu_i} \cdot \mu_i$$

since $\|z\| < 1$.

We now consider the cost of the commodity vectors y_i. We cannot estimate these costs individually, but the sum is easy to estimate. We obtain

$$\begin{aligned}
\sum \pi(y_i) &= \pi\left(\sum y_i\right) \\
&= \pi\left(\sum\left(x_i + \frac{1}{k\mu_i} v_i - \frac{1}{k}(a_i - b_i)\right)\right) \\
&= \sum \pi(x_i) + \frac{1}{k}\sum \frac{\pi(v_i)}{\mu_i} - \frac{1}{k}\pi(z).
\end{aligned}$$

Since $\pi(z) > \sum(\pi(v_i)/\mu_i)$. it follows that $\sum \pi(y_i) < \sum \pi(x_i)$, so that $\pi(y_j) < \pi(X_j) = \pi(e_j)$ for at least one agent j. Since $y_j \in P_j(x_1, \dots, x_N)$, this violates the assumption that $(x_1, \dots, x_N; \pi)$ is a quasi-equilibrium. Since we have obtained a contradiction to our supposition that $\sum(\pi(x_i)/\mu_i) < 1$, the proof is complete. □

Remark. For some Banach lattices L, we can actually do better. For example, if L is the Lebesgue space L_1, we can show that $\pi(v_i)/\mu_i \geqq 1$ for some agent i. However, in the general framework, the weaker conclusion of the Price Lemma is the most that is obtainable.

We stress that the Price Lemma depends crucially on the facts that L is a lattice and that the norm on L is a lattice norm, and may fail to be true if these assumptions are not satisfied. For example, the Price Lemma may fail if L is the two-dimensional vector lattice $\mathbb{R}^2$, equipped with a vector-space norm which is not a lattice norm.

6. Finite-dimensional vector sublattices

The object of this section is to prove that a Banach lattice which admits a compatible topology necessarily has a large collection of finite-dimensional vector sublattices. (A *vector sublattice* K of L is a linear subspace which is also a sublattice; we say K is a *finite-dimensional vector sublattice* if it is a vector sublattice and is finite-dimensional as a vector space.) We isolate the precise property we need in the following result:

Theorem 6.1. Let L be a Banach lattice which admits a compatible topology. Let $e_1, e_2, \ldots e_N$, $b_1, b_2, \ldots, b_M$ *be positive elements of L such that* $b_j \leqq \sum_{i=1}^N e_i$ *for each j, and let* $\delta > 0$ *be a positive number. Then there is a finite-dimensional vector sublattice K of L and there are positive elements* $e_1^*, \ldots, e_N^*$, $b_1^*, \ldots, b_M^*$ *of K such that*

(1) $0 \leqq e_i^* \leqq e_i$ *and* $\|e_i^* - e_i\| < \delta$ *for each i,*
(2) $0 \leqq b_j^* \leqq b_j$ *and* $\|b_j^* - b_j\| < \delta$ *for each j,*
(3) $\sum_{i=1}^N e_i^*$ *is strictly positive in K.*

It is convenient to first isolate a Lemma. Recall that L is *order complete* if every subset of L^+ which has an upper bound in L actually has a supremum in L.

Lemma 6.2. If the Banach lattice L admits a compatible topology, then L is order complete.

Proof Let Λ be any indexing set and let $\{z_\lambda\}_\Lambda$ be a family of positive elements of L bounded above by the positive element z. Let $\mathscr{F}$ be the set of finite subsets of Λ; for each F in $\mathscr{F}$, set $z_F = \sup\{z_\lambda : \lambda \in F\}$. Since $\mathscr{F}$ is directed by inclusion, the family $\{z_F : F \in \mathscr{F}\}$ is a net of positive elements of L; moreover, $z_F \leqq z$ for each F, so $\{z_F\}$ is a net in the order interval $[0, z]$. By assumption, L admits a Hausdorff vector space topology τ in which the order interval $[0, z]$ is compact. Hence some subnet $\{z_G : G \in \mathscr{J}\}$ converges (in the topology τ) to some element $\bar{z}$ of $[0, z]$. We assert that $\bar{z} = \sup\{z_F\}$.

To see this, we fix an element F_0 of $\mathscr{F}$. The definition of the elements z_F, together with the fact that $\{z_G\}$ is a subnet of $\{z_F\}$, imply that $\{z_G : z_G \geqq z_{F_0}\}$ is a subnet of $\{z_G]$, and hence also converges (in the topology τ) to $\bar{z}$. Since τ is a vector space topology, this implies that $\{z_G - z_{F_0}\}$ converges (in the topology τ) to $\bar{z} - z_{F_0}$. Since $z_G - z_{F_0}$ lies in the τ-compact order interval $[0, z - z_{F_0}]$, so does $\bar{z} - z_{F_0}$. In particular, $\bar{z} \geqq z_{F_0}$ for each F_0 in $\mathscr{F}$.

To see that $\bar{z} = \sup\{z_F\}$ we consider any w in L such that $w \geqq z_F$ for each F; we must show that $w \geqq \bar{z}$. Since $w \geqq z_F$ for each F, it follows in particular that $w \geqq z_G$ for each G in $\mathscr{J}$ and hence (as above) that $w \geqq \bar{z}$, as desired. Hence $\bar{z} = \sup\{z_F\}$, as asserted.

Finally, since $z_F = \sup\{z_\lambda : \lambda \in F\}$, it is clear that $\sup\{z_F\} = \sup\{z_\lambda : \lambda \in \Lambda\}$, so that $\{z_\lambda\}$ has a supremum. This completes the proof of Lemma 6.2. □

We now turn to the proof of Theorem 6.1.

Proof of Theorem 6.1. We may assume without loss that $0 < \delta < 1$. We set $e = \sum_{i=1}^N e_i$, and consider the *principal order ideal*

$$L_e = \{y \in L: -re \leqq y \leqq re \quad \text{for some integer } r\}.$$

According to Schaefer (1974, pp. 102, 104), L_e is an abstract M-space with e as order unit. and hence is order-isomorphic to the space $C(\Omega)$ of continuous real-valued functions on some compact Hausdorff space Ω. Moreover, under this isomorphism, the element e in L_e corresponds to the function on Ω which is identically equal to 1. Since L is order-complete, so is L_e. Hence by Schaefer (1974, p. 108) the space Ω is *Stonian*; i.e., the closure of every open subset of Ω is open.

In what follows, it will be convenient to suppress the isomorphism between L_e and $C(\Omega)$, and simply identify them. We will thus write $e_i(w)$ for the value of e_i at w, etc. We continue, however, to write $\|e_i\|$ for the norm of e_i in L, etc.

Choose any δ' with $0 < \delta' < \min(1/N, \delta/\|e\|)$. Since Ω is compact and each function e_i, b_j is continuous (hence uniformly continuous) we can find a covering of Ω by open sets $U_1, \dots, U_k$ such that, for each i, j, $|e_i(w) - e_i(w')| < \delta'$ and $|b_j(w) - b_j(w')| < \delta'$ whenever w, w' belong to the same set U_l. Let $\bar{U}_l$ denote the closure of u_l, and set $V_1 = \bar{U}_1$, $V_2 = \bar{U}_2 - V_1$, $V_3 = \bar{U}_3 - (V_1 \cup V_2)$, etc. Since Ω is Stonian, the sets $V_1, \dots, V_k$ form a cover of Ω by open and closed sets. (We may, without loss, assume that $V_l \neq 0$ for each l.) Moreover, for each i, j, $|e_i(w) - e_i(w')| \leqq \delta'$ and $|b_j(w) - b_j(w')| \leqq \delta'$ whenever w, w' belong to the same set V_l.

Now define K to be the subspace of $C(\Omega) = L_e$ consisting of functions which are constant on each of the sets V_l. It is evident that K is a finite-dimensional vector sublattice of $C(\Omega) = L_e$ (and hence of L). In fact, a basis for K consists of the characteristic functions χ_{V_l}, $l = 1, 2, \dots, k$.

For each i, j, l, let c_{il} be the minimum of the continuous function e_i on the compact set V_l, and let d_{jl} be the minimum of b_j on V_l. Set

$$e_i^* = \sum_{l=1}^{k} c_{il}\chi_{V_l},$$

$$b_j^* = \sum_{l=1}^{k} d_{jl}\chi_{V_l}.$$

This construction guarantees that the functions e_i^*, b_j^* are positive and satisfy the following inequalities for each w in Ω:

$$e_i^*(w) \leqq e_i(w) \quad \text{and} \quad b_j^*(w) \leqq b_j(w),$$

$$|e_i^*(w) - e_i(w)| \leqq \delta' \quad \text{and} \quad |b_j^*(w) - b_j(w)| \leqq \delta'.$$

The first two inequalities imply that $0 \leqq e_i^* \leqq e_i$ and $0 \leqq b_j^* \leqq b_j$. The second two inequalities, together with the fact that $e(w)=1$ for each w, imply that $|e_i^* - e_i| \leqq \delta' e$ and $|b_j^* - b_j| \leqq \delta' e$. By the lattice property of the norm (and the fact that $\delta' \|e\| < \delta$) this yields

$$\|e_i^* - e_i\| < \delta \quad \text{and} \quad \|b_j^* - b_j\| < \delta.$$

It remains only to show that $\sum e_i^*$ is strictly positive in K. Equivalently, we must show that for each l, at least one of the coefficients c_{il} is strictly positive. Fix a point w_l in V_l. Since $e(w_l)=1$, there is at least one e_i such that $e_i(w_l) \geqq 1/N$. Since the variation of e_i on V_l is at most δ', this means that $c_{il} \geqq (1/N) - \delta' > 0$, as required. This completes the proof of Theorem 6.1. □

7. Proof of the Main Existence Theorem

We begin by isolating parts of the argument as lemmas. The first one will be useful elsewhere, so we establish an appropriately general version; it is closely related to a finite-dimensional result of McKenzie (1959).

Lemma 7.1. Let $\mathcal{E} = \{(X_i, P_i, e_i)\colon i=1,2,\ldots,N\}$ be an irreducible economy in the Banach lattice L. Assume that $X_i = L^+$ for each i, and that the preference relation P_i is (norm, norm) continuous for each i. If $((x_1,\ldots,x_n),\pi)$ is a quasi-equilibrium for $\mathcal{E}$ and there is a vector $z \in L$ such that $0 \leqq z \leqq \sum e_i$ and $\pi(z) \neq 0$, then $(x_1,\ldots,x_N,\pi)$ is actually an equilibrium.

Proof. Let I denote the set of agents i for which there is a vector ζ with $0 \leqq \zeta \leqq e_i$ and $\pi(\zeta) \neq 0$; let J denote the complementary set of agents. We first show that the equilibrium conditions are satisfied for all agents in I.

Fix $i \in I$ and a vector ζ with $0 \leqq \zeta \leqq e_i$ and $\pi(\zeta) \neq 0$. Let $y_i \in P_i(x_1, \ldots, x_N)$; then $\pi(y_i) \geqq \pi(e_i)$, and we want to show that in fact $\pi(y_i) > \pi(e_i)$. If $\pi(y_i) = \pi(e_i)$, we distinguish three cases. *Case 1.* $\pi(e_i) > 0$. Then $\pi(ty_i) < \pi(e_i)$ for $t < 1$, while $ty_i \in P_i(x_1, \ldots, x_N)$ if t is close to 1 (by continuity of P_i). This violates the quasi-equilibrium conditions. *Case 2.* $\pi(e_i) < 0$. Then $\pi(sy_i) < \pi(e_i)$ for $s > 1$ and $sy_i \in P_i(x_1, \ldots, x_N)$ for s near 1, again violating the quasi-equilibrium conditions. *Case 3.* $\pi(e_i) = 0$. By continuity of P_i, for all small real numbers $r > 0$ we have $y_i + re_i \in P_i(x_1, \ldots, x_N)$. Since $0 \leqq \zeta \leqq e_i$, we know that $y_i + re_i + r^*\zeta \in P_i(x_1, \ldots, x_N)$ provided that $|r^*|$ is sufficiently small (if $|r^*| < r$ then $y_i + re_i + r^*\zeta \geqq 0$). On the other hand, $\pi(y_i + re_i + r^*\zeta) = r^*\zeta) = r^*\pi(\zeta)$, and $r^*\pi(\zeta) < 0 = \pi(e_i)$ if r^* and $\pi(\zeta)$ have opposite signs. This again violates the quasi-equilibrium conditions. We conclude that the equilibrium conditions hold for all agents in I.

Notice that I is not empty. For, since $0 \leqq z \leqq \sum e_i$, we may use the Riesz Decomposition Property to write $z = \sum z_i$ with $0 \leqq z_i \leqq e_i$ for each i. Then $\pi(z) = \sum \pi(z_i) \neq 0$, so $\pi(z_i) \neq 0$ for at least one agent i, and this agent belongs to I.

Finally we show that J is empty. For, if not, irreducibility of $\mathcal{E}$ guarantees that there is an $i \in I$, a $j \in J$ and a $\zeta \in L$ such that $0 \leqq \zeta \leqq e_j$ and $x_i + \zeta \in P_i(x_1, \ldots, x_N)$. Since $j \in J$ we know that $\pi(\zeta) = 0$, so that $\pi(x_i + \zeta) = \pi(x_i) \leqq \pi(e_i)$; this violates the equilibrium conditions just established for agent i. We conclude that J is empty, and hence that $(x_1, \ldots, x_N, \pi)$ is an equilibrium. This completes the proof of Lemma 7.1. □

Throughout the remainder of this section, we assume that all the hypotheses of the Main Existence Theorem are satisfied (except for irreducibility of the economy $\mathcal{E}$).

We are going to obtain a quasi-equilibrium for $\mathcal{E}$ as a limit of equilibria of subeconomies whose commodity spaces are finite-dimensional vector sublattices of L. Because the family of finite-dimensional vector sublattices L is not directed by inclusion, we need to carry along some extra information. The precise structure we need is an $(N+2)$-tuple $\alpha = (F_\alpha, n_\alpha, e_1^\alpha, \ldots, e_N^\alpha)$, where F_α is a finite-dimensional vector sublattice of L, n_α is a positive integer and $e_1^\alpha, \ldots, e_N^\alpha$ are positive elements of F_α which satisfy

(a) $\sum_{i=1}^N e_i^\alpha$ is strictly positive in F_α,
(b) $\|e_i - e_i^\alpha\| < 1/n_\alpha$ for each i,
(c) $e_i^\alpha \leqq e_i$ for each i.

We shall call such an $(N+2)$-tuple α a *special configuration.* (Notice that, since F_α is a vector sublattice of L, it follows that $\sum e_i^\alpha$ is a positive element of L, but it need not be strictly positive in L.) The vectors e_i^α will be the initial endowments of our approximating subeconomies. The integers n_α will play a role when we make the family of all special configurations into a directed set.

Our first task is to show that many special configurations exist. Given a finite subset A of L^+ we cannot generally find a special configuration $\alpha = (F_\alpha, n_\alpha, e_1^\alpha, \dots, e_n^\alpha)$ with $A \subset F_\alpha$, but we can come as close as we wish.

Lemma 7.2 Let A be a finite subset of L^+ and $\varepsilon > 0$ a positive number. Then there is a special configuration $\alpha = (F_\alpha, n_\alpha, e_1^\alpha, \dots, e_N^\alpha)$ such that $1/n_\alpha < \varepsilon$ and

$$\operatorname{dist}(a, F_\alpha^+) = \inf\{\|a - z\| : z \in F_\alpha^+\} < \varepsilon$$

for each element a of A.

Proof. Write $A = \{a_1, \dots, a_M\}$, and choose an integer s with $1/s < \varepsilon$. Since e is strictly positive, $\lim_{n\to\infty}(ne \wedge a_j) = a_j$ (for each j). Hence we can find an integer R so large that

$$\|(Re \wedge a_j) - a_j\| < \varepsilon/2$$

for each j. Set $b_j = R^{-1}(Re \wedge a_j)$ for each j, and note that $0 \leqq b_j \leqq e$ for each j. Hence, we may set $\delta = \min(1/s, \varepsilon/2M)$ and apply Theorem 6.1 to obtain a finite-dimensional vector sublattice K of L and elements $e_1^*, \dots, e_N^*$, $b_1^*, \dots, b_M^*$ of K such that

(1) $0 \leqq e_i^* \leqq e_i$ and $\|e_i^* - e_i\| < \delta$ for each i,
(2) $0 \leqq b_j^* \leqq b_j$ and $\|b_j^* - b_j\| < \delta$ for each j,
(3) $\sum_{i=1}^N e_i^*$ is strictly positive in K.

Then $\alpha = (K, s, e_1^*, \dots, e_N^*)$ is a special configuration. Moreover, for each j, Rb_j^* belongs to K and our choice of b_j and the triangle inequality imply that $\|a_j - Rb_j^*\| < \varepsilon$. Hence the special configuration α has the required properties, and the proof is complete. □

We will write D for the set of all special configurations. We wish to use D to index nets of quasi-equilibria; to do so, we must define an ordering on D.

Given two special configurations $\alpha = (F_\alpha, n_\alpha, e_1^\alpha, \dots, e_N^\alpha)$ and $\beta = (F_\beta, n_\beta, e_1^\beta, \dots, e_N^\beta)$, we will write $\alpha < \beta$ provided that $n_\alpha < n_\beta$ and

$$\operatorname{dist}(z, F_\beta^+) \leqq 2^{-n_\alpha}\|z\|$$

for each $z \in F_\alpha^+$. This relation is not transitive, but it is acyclic; i.e., there is no finite sequence $\alpha_1, \alpha_2, \dots, \alpha_k$ of special configurations such that $\alpha_1 < \alpha_2 < \cdots < \alpha_k < \alpha_1$ (because we cannot have $n_{\alpha_1} < n_{\alpha_2} \cdots < n_{\alpha_k} < n_{\alpha_1}$). Hence this relation can be extended[7] to a reflexive, antisymmetric, transitive relation $\leqq$ on D (i.e., a *partial ordering*) which is given by

$\alpha \leqq \beta$ if either (i) $\alpha = \beta$, or (ii) there are elements $\gamma_1, \dots, \gamma_k$ of D such that

$$\alpha = \gamma_1, \quad \beta = \gamma_k \quad \text{and} \quad \gamma_1 < \gamma_2 \cdots < \gamma_k.$$

A simple calculation, using (ii), the triangle inequality and the fact that $\sum_{n=1}^{\infty} 2^{-n} = 1$, shows that if $\alpha \leqq \beta$ then

$$\operatorname{dist}(z, F_\beta^+) \leqq 4 \cdot s^{-n_\alpha} \|z\|$$

for each $z \in F_\alpha^+$. We next show that the partial ordering $\leqq$ actually directs the set D.

Lemma 7.3. The set D of special configurations, equipped with the partial ordering $\leqq$, is a directed set. That is, if α and β belong to D then there is a γ in D for which $\alpha \leqq \gamma$ and $\beta \leqq \gamma$.

Proof. Since F_α and F_β are finite-dimensional, their unit spheres are compact. Hence we can choose finite sets of vectors $\{x_1, \dots, x_J\} \subset F_\alpha$ and $\{y_1, \dots, y_K\} \subset F_\beta$ such that $\|x_j\| = 1$ for each j; $\|y_k\| = 1$ for each k; for each $x \in F_\alpha$ with $\|x\| = 1$, there is an index j with $\|x - x_j\| < 2^{-2-n_\alpha}$ and for each $y \in F_\beta$, with $\|y\| = 1$, there is an index k with $\|y - y_k\| < 2^{-2-} n^\beta$. We now use Lemma 7.2 to choose a special configuration γ such that $1/n_\gamma < 1/(n_\alpha + n_\beta)$ (which means $n_\gamma > n_\alpha + n_\beta$), $\operatorname{dist}(x_j, F_\gamma^+) < 2^{-n_\alpha - 2}$ for each j and $\operatorname{dist}(y_k, F_\gamma^+) < 2^{-n_\beta - 2}$ for each k. Since the norm of F_γ is positively homogeneous, the triangle inequality and our choice of $\{x_1, \dots, x_J\}$ and $\{y_1, \dots, y_K\}$ imply that $\operatorname{dist}(x, F_\gamma^+) \leqq 2^{-n_\alpha} \|x\|$ for each $x \in F_\alpha^+$ and $\operatorname{dist}(y, F_\gamma^+) \leqq 2^{-n_\beta} \|y\|$ for each $y \in F^{+\beta}$. Thus $\alpha < \gamma$ and $\beta < \gamma$; in particular, $\alpha \leqq \gamma$ and $\beta \leqq \gamma$, as desired. □

With all of the preliminary constructions out of the way, we now turn to the main argument.

Proof of the Main Existence Theorem. For each special configuration $\alpha = (F_\alpha, n_\alpha, e_1^\alpha, \dots, e_N^\alpha)$ we define consumption sets X_i^α and preference relations $p_i^\alpha \colon \prod X_j^\alpha \to 2^{X_i^\alpha}$ by

[7]Any acyclic relation may always be extended to a reflexive, antisymmetric transitive relation by exactly this procedure.

$$X_i^\alpha = X_i \cap F_\alpha = F_\alpha^+,$$

$$P_i^\alpha(x_1, \dots, x_N) = P_i(x_1, \dots, x_N) \cap F_\alpha.$$

We set $\mathcal{E}^\alpha = \{(X_i^\alpha, P_i^\alpha, e_i^\alpha)]$; this is an economy in the finite-dimensional vector sublattice F_α of L. The next step is to find extemely desirable commodities.

For each i, fix a vector $v_i \in L^+$ which is extremely desirable (on the set $\mathscr{A}$) for consumer i. For each α, the distance from v_i to F_α^+ is acutally taken on (since F_α is finite-dimensional); i.e., we can choose vectors v_i^α in F_α so that

$$\|v_i^\alpha - v_i\| = \inf\{\|z - v_i\| : z \in F_\alpha^+\}.$$

For any $\varepsilon > 0$, we can use Lemma 7.2 to find a special configuration α such that $\|v_i^\alpha - v_i\| < \varepsilon$ for each i. On the other hand, the properties of our ordering require that $\operatorname{dist}(v_i^\alpha, F_\beta^+) \leqq 4 \cdot 2^{-n_\alpha} \|v_i^\alpha\| \leqq 4 \cdot 2^{-n_\alpha}(\|v_i\| + \varepsilon)$. whenever $\beta \geqq \alpha$. Hence there is a vector in F_β^+ whose distance to v_i is at most $4 \cdot 2^{-n_\alpha}(\|v_i\| + \varepsilon) \times \varepsilon$. The definition of v_i^β now yields that

$$\|v_i^\beta - v_i\| \leqq 4 \cdot 2^{-n_\alpha}(\|v_i\| + \varepsilon) + \varepsilon$$

whenever $\beta \geqq \alpha$. We conclude that, for each i, $\|v_i^\gamma - v_i\|$ tends to 0 along the directed set D of special configurations. It follows immediately from Lemma 3.2 that (for each i), v_i^γ is extremely desirable for consumer i (on the set A), provided γ is sufficiently large. Since v_i^γ bellongs to F_γ^+, it is certainly extremely desirable for consumer i on $A \cap F_\gamma^+$, which includes the set of feasible allocations for the economy $\mathcal{E}^\gamma$ (provided that γ is sufficiently large).

We now want to apply the equilibrium existence result of Shafer (1976, Theorem 2 and Remarks) to conclude that each of the economies $\mathcal{E}^\gamma$ has a quasi-equilibrium $(x_1^\gamma, \dots, x_N^\gamma; p^\gamma)$, for γ sufficiently large. To do so, we first note that the existence of extremely desirable commodities implies that preferences are locally non-satiated on the set of feasible allocations. Shafer's continuity assumptions follow from continuity of the preferences P_i, together with the fact that all Hausdorff vector space topologies on a finite-dimensional vector space coincide. The remaining conditions of Shafer's Theorem are easily verified, except for the requirement that the initial endowments lie in the interiors of the consumption sets. To remedy this small difficulty, we choose a real number t with $0 < t < 1$ and define new endowments

$$f_i^t = (1-t)e_i^\gamma + (t/N) \sum_{j=1}^{N} e_j^\gamma.$$

These new endowments do lie in the interiors of the consumption sets, so the economy with these endowments has an equilibrium (note that $\sum f_i^t = \sum e_i^\gamma$). Letting t tend to zero and taking limits yields our quasi-equilibrium $(x_1^\gamma, \ldots, x_N^\gamma; p^\gamma)$. The price p^γ belongs to F'_γ, and there is no loss in assuming that $\|p^\gamma\| = 1$. The Hahn–Banach Theorem now provides an element π^γ of L' which has norm one, and agrees with p^γ on F_γ.

We have thus constructed a net $\{(x_1^\gamma, \ldots, x_N^\gamma; \pi^\gamma)\}$ in $L \times L \cdots \times L \times L'$. Since $0 \leqq \sum x_i^\gamma = \sum e_i^\gamma \leqq e$ for each γ, the vectors x_i^γ all belong to the order interval $\{0, e]$, which is τ-compact. Moreover, since $\|\pi^\gamma\| = 1$, the functionals π^γ all belong to the unit ball of L', which is weak-star compact. Hence, passing to a subnet if necessary, we obtain vectors $\bar{x}_1, \ldots, \bar{x}_N$ in $(0, e]$ and a functional $\bar{\pi}$ in L' such that $\{x_i^\gamma\}$ converges to $\bar{x}_i$ (in the topology τ) and $\{\pi^\gamma\}$ converges to $\bar{\pi}$ (in the weak-star topology). Note that $\|\bar{\pi}\| \leqq 1$.

We are now going to show that $(\bar{x}_1, \ldots, \bar{x}_N; \bar{\pi})$ is a quasi-equilibrium for $\mathcal{E}$. Our first task is to show that $\bar{\pi}$ is not the zero functional; as we have emphasized, this is the crucial point. To do this, we first recall that $\|v_i^\gamma - v_i\|$ tends to zero, for each i, so that $\{\pi^\gamma(v_i^\gamma)\}$ converges to $\bar{\pi}(v_i)$, for each i (by Lemma A of the appendix). The Price Lemma gives us an estimate involving the $\pi^\gamma(v_i^\gamma)$, namely

$$\sum_{i=1}^{N} \frac{\pi^\gamma(v_i^\gamma)}{\mu_i(v_i^\gamma, (x_1^\gamma, \ldots, x_N^\gamma))} \geqq 1 \tag{*}$$

provided that we compute marginal rates of desirability in the economy $\mathcal{E}^\gamma$. However, let us note that $\pi^\gamma(v_i^\gamma) \geqq 0$ (since v_i^γ is extremely desirable and π^γ is a quasi-equilibrium price) and that marginal rates of desirability certainly do not increase if we compute them in $\mathcal{E}$ rather than in $\mathcal{E}^\gamma$. Hence the inequality (*) is valid if we compute marginal rates of desirability in the economy $\mathcal{E}$. Let us write

$$\mu_i = \inf\{\mu_i(v_i, (x_1, \ldots, x_N)) : (x_1, \ldots, x_N) \in \mathscr{A}\}.$$

By extreme desirability, $\mu_i > 0$. By Lemma 4.3,

$$\mu_i(v_i^\gamma, (x_1^\gamma, \ldots, x_N^\gamma)) \geqq \mu_i - \|v_i - v_i^\gamma\|.$$

As we have already noted, $\|v_i - v_i^\gamma\|$ tends to 0. If we combine this fact with the inequality (*) and our previous observation that $\{\pi^\gamma(v_i^\gamma)\}$ converges to $\bar{\pi}(v_i)$, we obtain

$$\sum_{i=1}^{N} \frac{\bar{\pi}(v_i)}{\mu_i} \geqq 1.$$

In particular, $\bar{\pi}$ is not the zero functional.

We now proceed to verify the quasi-equilibrium conditions. The argument is similar to Bewley's (1974), but more complicated, since the endowments e_i^γ may differ from the endowments e_i. First of all, we know that $\sum x_i^\gamma = \sum e_i^\gamma$ for each γ. By construction. the endowments e_i^γ converge to e_i in the norm topology and hence in the topology τ (which is weaker than the norm topology). Since the vectors x_i^γ converge to $\bar{x}_i$ in the topology τ, and τ is a vector space topology we conclude that $\sum \bar{x}_i = \sum e_i$.

Now let us suppose that $y_i \in P_i(\bar{x}_1, \ldots, \bar{x}_N)$ snd that $\bar{\pi}(y_i) < \bar{\pi}(e_i)$. Proceeding exactly as in the construction of extremely desirable commodities, we find vectors $y_i^\gamma \in F_\gamma^+$ such that $\|y_i^\gamma - y_i\|$ tends to 0 (with γ). Since the preference relation P_i is (τ, norm)-continuous, we conclude that $y_i^\gamma \in P_i(x_1^\gamma, \ldots, x_N^\gamma)$ if γ is large enough. Since the vectors in question all belong to F_γ, it follows that $y_i^\gamma \in P_i^\gamma(x_1^\gamma, \ldots, x_N^\gamma)$. On the other hand, since $\|y_i^\gamma - y_i\|$ and $\|e_i^\gamma - e_i\|$ both tend to 0 (with γ), and $\{\pi^\gamma\}$ converges to $\bar{\pi}$ in the weak-star topology, we may apply Lemma A of the appendix again to conclude that $\pi^\gamma(y_i^\gamma) < \pi^\gamma(e_i^\gamma)$ for γ sufficiently large. Since $y_i^\gamma \in P_i^\gamma(x_1^\gamma, \ldots, x_N^\gamma)$ for large γ, this contradicts the fact that $(x_1^\gamma, \ldots, x_N^\gamma, \pi^\gamma)$ is a quasi-equilibrium for $\mathcal{E}^\gamma$. We conclude that, if $y_i \in P_i(\bar{x}_i, \ldots, \bar{x}_N)$, then $\bar{\pi}(y_i) \geqq \bar{\pi}(e_i)$.

Finally, we need to show that $\bar{\pi}(\bar{x}_i) = \bar{\pi}(e_i)$ for each i. But if $\bar{\pi}(\bar{x}_i) \neq \bar{\pi}(e_i)$ for some i, we must have $\bar{\pi}(\bar{x}_j) < \bar{\pi}(e_j)$ for some j, since $\sum \bar{x}_i = \sum e_i$. On the other hand, $\bar{x}_j + tv_j \in P_j(\bar{x}_1, \ldots, \bar{x}_N)$ for each $t > 0$ (since v_j is extremely desirable) and $\bar{\pi}(\bar{x}_j + tv_j) < \bar{\pi}(e_j)$ for t sufficiently small (since $\bar{\pi}(x_j) < \bar{\pi}(e_j)$). This contradicts the conclusion of the previous paragraph. This completes the proof that $(\bar{x}_1, \ldots, \bar{x}_N, \bar{\pi})$ is a quasi-equilibrium.

It remains to show that, when $\mathcal{E}$ is irreducible, every quasi-equilibrium $(x_1^*, \ldots, x_N^*; \pi^*)$ is actually an equilibrium. Since π^* is a non-zero price, $\pi^*(z) \neq 0$ for some positive z. Since e is strictly positive, $\|(ne \wedge z) - z\|$ is small provided n is large. Hence $\pi^*(ne \wedge z) \neq 0$, so that $\pi^*(1/n)(ne \wedge z) \neq 0$ while $(1/n)(ne \wedge z) \leqq e$; Lemma 7.1 now implies that $(x_1^*, \ldots, x_N^*, \pi^*)$ is actually an equilibrium. This completes the proof. □

It is worth noting that, by the same argument, we can show that the set of all quasi-equilibria $(\bar{x}_1, \ldots, \bar{x}_N; \bar{\pi})$ with $\|\pi\| \leqq 1$ is a compact subset of $L \times L \cdots \times L \times L'$, where we give L the topology τ and L' the weak-star topology. (Of course, if $\mathcal{E}$ is irreducible, the set of equilibria is compact, since it coincides with the set of quasi-equilibria.)

8. Concluding remarks

Remark 1. Throughout, we have assumed that the consumption set of each agent is the positive cone L^+. An examination of the proof will show, however, that it works equally well for (some) other consumption sets. For example, it would suffice to assume that the consumption set X_i of the ith agent has the properties:

(a) X_i is a closed, convex subset of L^+ containing 0,
(b) X_i is solid (i.e., if $x \in X_i$ then the order interval $[0, x]$ is contained in X_i),
(c) if $x \in X_i$ then $x + tv_i \in X_i$ for some $t > 0$.

Remark 2. Much of our analysis should go through in the context of a countable number of agents, provided the commodity space is l_1 or L_1. For more general commodity spaces, there seem to be additional serious difficulties. (See the Remark in section 5.)

Remark 3. Notice that, in the proofs of the Main Existence Theorem, the Price Lemma and Theorem 6.1, completeness of the norm of L was never used. The Main Existence Theorem therefore remains valid for incomplete normed vector lattices.

Remark 4. As we noted in section 4, the requirement that the aggregate initial endowment be strictly positive rules out the commodity space $M(\Omega)$, which has no strictly positive elements. However, our results can be extended to this case, if we strengthen the extreme desirability assumption slightly. Here is a sketch:

We will assume that for each consumer i, there is a commodity v_i which belongs to the order interval $[0, e]$ and is extremely desirable for i on some open set containing $[0, e]^N$. For each finite set $A = \{a_1, \ldots, a_k\}$ contained in $M(\Omega)^+$, we write $f_A = e + \sum a_i$, and consider the set M_A of measures which are absolutely continuous with respect to f_A; this is a closed sublattice of $M(\Omega)$ and f_A is strictly positive when viewed in the sublattice M_A. If we consider the restriction of $\mathcal{E}$ to M_A, and alter the initial endowment of each consumer to be $\tilde{e}_i = e_i + \varepsilon f_A$, for ε a small positive number, then we obtain an economy $\mathcal{E}_{A,\varepsilon}$ to which the Main Existence Theorem may be applied. The economies $\mathcal{E}_{A,\varepsilon}$ thus have quasi-equilibria; moreover, if π is a quasi-equilibrium price then $\sum(\pi(v_i)/\mu_i) \geqq 1$. If we now take limits (as A increases and ε tends to 0), we obtain a quasi-equilibrium $(\bar{x}_1, \ldots, \bar{x}_N; \bar{\pi})$ for $\mathcal{E}$ with $\sum(\bar{\pi}(v_i)/\mu_i) \geqq 1$. Hence, if $\mathcal{E}$ is irreducible, Lemma 7.2 can again be used to prove that $(\bar{x}_1, \ldots, \bar{x}_N; \bar{\pi})$ is an equilibrium.

Note that this argument produces an equilibrium price π in $M(\Omega)'$, not in $C(\Omega)$. If we want the price to lie in $C(\Omega)$, we must assume much more; see Yannelis–Zame (1984) for details.

Appendix

In this appendix we collect some basic information about Banach spaces in general and Banach lattices in particular. For further details, we refer the reader to Schaefer (1971, 1974).

A *normed vector space* is a real vector space E equipped with a norm $\|\cdot\|: E \to [0, \infty)$ satisfying:

(i) $\|x\| \geqq 0$ for all x in E, and $\|x\| = 0$ if and only if $x = 0$,
(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all x in E and all α in $\mathbb{R}$,
(iii) $\|x + y\| \leqq \|x\| + \|y\|$ for all x, y in E.

The *Banach space* is a normed vector space for which the metric induced by the norm is complete.

If E is a Banach space, then its *dual space* E' is a set of continuous linear functionals on E. The dual space E' is itself a Banach space, when equipped with the norm

$$\|\phi\| = \sup\{|\phi(x)| : x \in E, \|x\| \leqq 1\}.$$

In addition to the norm topologies on E and E', we shall make use of three other topologies. The *weak topology* $\sigma(E, E')$ on E is the topology of pointwise convergence when we regard elements of E as functionals on E'. That is, a net $\{x_\alpha\}$ in E converges weakly to an element $x \in E$ exactly when $\{f(x_\alpha)\}$ converges to $f(x)$ or each $f \in E'$. Similarly, the *weak-star topology* $\sigma(E', E)$ on E' is the topology of pointwise convergence when we regard elements of E' as functionals on E. Thus $f_\alpha \to f$ in the weak-star topology means that $f_\alpha(x) \to f(x)$ for each $x \in E$. Finally, the *Mackey topology* $\tau(E', E)$ on E' is the topology of uniform convergence on weakly compact, convex, symmetric, subsets of E.

It is a consequence of the Separation Theorem that the weak and norm topologies on E have the same closed convex sets and the same continuous linear functionals. The Mackey–Arens Theorem asserts that the weak-star and Mackey topologies on E' have the same closed convex sets and the same continuous linear functionals, and that the Mackey topology is the strongest locally convex vector space topology on E' with this property. By viewing elements of E as linear functionals on E' we obtain a canonical injection of E into E'' and we may identify E as the subspace of weak-star continuous linear functionals on E'.

Alaoglu's Theorem asserts that the closed unit ball of E' (and hence every weak-star closed, norm bounded set) is weak-star compact. Hence every net $\{\pi_\alpha\}$ in the ball of E' has a convergent/subnet. As a final comment, let us note for further use the following elementary lemma:

Lemma A. If $x_\alpha \to x$ in the norm topology of E, $\pi_\alpha \to \pi$ in the weak-star topology of E' and $\{\pi_\alpha\}$ is norm bounded, then $\pi_\alpha(x_\alpha) \to \pi(x)$.

Recall that a *Banach lattice* is a Banach space L endowed with a partial order $\leqq$ (i.e., $\leqq$ is a reflexive, antisymmetric, transmitive relation) satisfying:

(1) $x \leqq y$ implies $x+z \leqq y+z$ (for all $x, y, z \in L$),
(2) $x \leqq y$ implies $tx \leqq ty$ (for all $x, y, \in L$, all real numbers $t \geqq 0$),
(3) every pair of elements $x, y \in L$ has a supremum (least upper bound) $x \vee y$ and an infimum (greatest lower bound) $x \wedge y$,
(4) $|x| \leqq |y|$ implies $\|x\| \leqq \|y\|$ (for all x, $y \in L$).

Here we have written, as usual, $|x| = x^+ + x^-$ where $x^+ = x \vee 0$, $x^- = (-x) \vee 0$; we call x^+, x^- the *positive* and *negative parts* of x (respectively) and $|x|$ the *absolute value* of x. We recall that $x = x^+ - x^-$, and that $x^+ \wedge x^- = 0$. We say that $x \in L$ is *positive* if $x \geqq 0$; we write L^+ for the set of all positive elements of L and refer to L^+ as the *positive cone* of L.

The Banach lattice structure on L induces on the dual space L' the structure of a Banach lattice, where $f \leqq g$ in L' means $f(x) < g(x)$ for each $x \in L'$.

If x is a positive element of L, then by the *order interval* $[0, x]$ we mean the set

$$[0, x] = [y: y \in L, 0 \leqq y \leqq x\}.$$

In any Banach lattice L, order intervals are norm closed (and thus weakly closed), convex and bounded. If L is a dual lattice, order intervals are also weak-star closed (and thus weak-star compact).

We shall say that an element x of L is *strictly positive* (and write $x \gg 0$) if $\phi(x) > 0$ whenever ϕ is a positive non-zero element of L^+. (Strictly positive elements are sometimes called *quasi-interior* to L^+.) An equivalent characterization is that the element x in L is strictly positive if and only if the sequence $\{nx \wedge y\}$ converges in norm to y (as n tends to infinity) for each y in L^+. We note that if the positive cone L^+ of L has a non-empty (norm) interior, then the set of strictly positive elements coincides with the interior of L^+. However, many Banach lattices contain strictly positive elements even though the positive cone L^+ has an empty interior.

Basic examples of Banach lattices include:

(i) the Euclidean space $\mathbb{R}^N$,
(ii) the space l_p (for $1 \leqq p < \infty$) of real sequences (a_n) for which the norm $\|(a_n)\|_p = (\sum |a_n|^p)^{1/p}$ is finite,
(iii) the space $L_p(\Omega, R, \mu)$ of measurable functions f on the measure space $(\Omega. R, m)$ for which the norm $\|f\|_p = (\int_\Omega |f|^p \, d\mu)^{1/p}$ is finite (as usual, we identify two functions if they agree almost everywhere),
(iv) the space l_∞ of bounded real sequences (with the supremum norm),
(v) the space $L_\infty(\Omega, R, m)$ of essentially bounded, measurable functions on a measure space [with the essential supremum norm, and the same identification as in (iii)].

(vi) the space $C(\Omega)$ of continuous, real-valued functions on the compact Hausdorff space Ω (with the supremum norm),
(vii) the space $M(\Omega)$ of regular Borel measures on the compact Hausdorff space Ω (with the total variation norm).

In examples (i), (ii), (iii), and (vi), a function (or N-tuple, or sequence) is strictly positive in the Banach lattice sense exactly when it is strictly positive as a function (almost everywhere). In (iv) and (v), a function is strictly positive in the Banach lattice sense exactly when it is positive and bounded away from zero. Finally, in (vii), a measure μ is strictly positive in the Banach lattice sense exactly when $\mu(B)>0$ for every Borel set B; thus if Ω is uncountable, $M(\Omega)$ contains no strictly positive elements.

A fundamental property of Banach lattices (actually valid more generally for vector lattices) which we shall use over and over, is the Riesz Decomposition Property.

Riesz Decomposition Property. Let L be a Banach lattice and let $x, y_1, \ldots, y_n$ *be positive elements of L such that* $0 \leqq x \leqq \sum_{i=1}^{n} y_i$. *Then there are positive elements* $x_1, \ldots, x_n$ in *L such that* $\sum_{i=1}^{n} x_i = x$ and $0 \leqq x_i \leqq y_i$ for each i.

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