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Equilibria in Markets with a Continuum of Agents and Commodities

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Abstract. We prove the existence of an equilibrium for an exchange economy with a measure space of agents and with an infinite dimensional commodity space.

1. Introduction

The purpose of this paper is to prove the existence of a competitive equilibrium for an economy with a measure space of agents and with an infinite dimensional commodity space.

The principle ways our result differs from that of Bewley (1990) are: (a) we assume that the consumption set of each agent is a weakly compact subset of either the space of continuous functions on a compact metric space $C(X)$, or the Lebesgue space L_∞; (b) the measure space of agents need not be atomless; and (c) we provide a direct proof, i.e., we do not need to use the Aumann (1966) existence result as Bewley does.

The paper proceeds as follows: Section 2 contains some notation and definitions. In Section 3 the main result of the paper is stated. An auxiliary result is stated in Section 4 and its proof is given in Section 5. Section 6 contains the proof of the main theorem. Finally some concluding remarks are given in Section 7.

2. Notation and Definitions

2.1 Notation.

2^A denotes the set of all nonempty subsets of the set A;
con A denotes the convex hull of the set A;
$\overline{\text{con}}A$ denotes the closed convex hull of the set A;
$\setminus$ denotes the set theoretic subtraction;
$\mathbb{R}^\ell$ denotes the ℓ-fold Cartesian product of the set of real numbers $\mathbb{R}$;
$\emptyset$ denotes the empty set.

2.2 Definitions. Let X, Y be two topological spaces. A set-valued function (or correspondence) $\phi : X \to 2^Y$ is said to be *upper semicontinuous* (u.s.c.) if the set $\{x \in X : \phi(x) \subset V\}$ is open in X for every open subset V of Y. Throughout the paper we will consider the setting where X is a metric space and Y is a Banach space. In this setting we will say that ϕ is *norm* u.s.c., if the set $\{x \in X : \phi(x) \subset V\}$ is open in X for every norm open subset V of Y. Furthermore, we will say that ϕ is *weakly* u.s.c., if the set $\{x \in X : \phi(x) \subset V\}$ is open in X for every weakly open subset V of Y.

Let X and Y be sets. The *graph* of the correspondence $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$.

We now define the notion of a Bochner integrable function. Let (T, τ, μ) be a finite measure space, and X be a Banach space. A function $f : T \to X$ is called *simple* if there exist $x_1, x_2, \ldots, x_n$ in X and $\tau_1, \tau_2, \ldots, \tau_n$ in τ such that $f = \sum_{i=1}^n x_i \chi_{\tau_i}$, where

$$\chi_{\tau_i}(t) = \begin{cases} 1 & \text{if } t \in \tau_i \\ 0 & \text{if } t \notin \tau_i. \end{cases}$$

A function $f : T \to X$ is said to be *μ-measurable* if there exists a sequence of simple functions $f_n : T \to X$ such that $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ for almost all $t \in T$. A μ-measurable function $f : T \to X$ is said to be *Bochner integrable* if there exists a sequence of simple functions $\{f_n : n = 1, 2, \ldots\}$ such that

$$\lim_{n\to\infty} \int_T \|f_n(t) - f(t)\| d\mu(t) = 0.$$

In this case we define for each $E \in \tau$ the integral to be $\int_E f(t) d\mu(t) = \lim_{n\to\infty} \int_E f_n(t) d\mu(t))$. It is a standard result [see Diestel-Uhl (1977, Theorem 2, p. 45)] that, if $f : T \to X$ is a μ-measurable function then ϕ is Bochner integrable if and only if $\int_T \|f(t)\| d\mu(t) < \infty$.

Let (T, τ, μ) be a complete finite measure space, i.e., μ is a real-valued, non-negative countably additive measure defined on a complete σ-field τ of subsets of T such that $\mu(T) < \infty$. Let X be a Banach space. We denote by $L_1(\mu, X)$ the space of equivalence classes of X-valued Bochner integrable functions $f : T \to X$ normed by

$$\|f\| = \int_T \|f(t)\| d\mu(t).$$

Normed by the functional $\|\cdot\|$ above, $L_1(\mu, X)$ becomes a Banach space [(see Diestel-Uhl (1977, p. 50)]. A correspondence $\phi: T \to 2^X$ is said to have a *measurable graph* if $G_\phi \in \tau \otimes \beta(X)$ where $\beta(X)$ denotes the Borel σ-algebra on X and $\otimes$ denotes product σ-algebra. The correspondence $\phi: T \to 2^X$ is said to be *lower measurable* if for every open subset V of X the set $\{t \in T : \phi(t) \cap V \neq \emptyset\}$ belongs to τ. The correspondence $\phi: T \to 2^X$ is said to be *integrably bounded* if there exists a map $h \in L_1(\mu, \mathbb{R})$ such that for almost all $t \in T$, $\sup\{\|x\| : x \in \phi(t)\} \leq h(t)$. A *measurable selection* for the correspondence $\phi: T \to 2^X$ is a measurable function $f: T \to X$ such that $f(t) \in \phi(t)$ for almost all $t \in T$. A well-known result of Aumann (1967) says that if ϕ is a correspondence from a complete finite measure space to a separable metric space such that ϕ has a measurable graph and it is nonempty valued, then ϕ has a measurable selection. Following Aumann (1965) we now define the notion of the Aumann integral. Let T be a finite measure space, X be a Banach space and $\phi: T \to 2^X$ be a correspondence. We denote by S^1_ϕ the set of all X-valued Bochner integrable selections for $\phi(\cdot)$, i.e., $S^1_\phi = \{x \in L_1(\mu, X) : x(t) \in \phi(t)$ for almost all $t \in T\}$. In the sequel we will call the above set, *the set of integrable selections.* We are now ready to define the integral of the correspondence $\phi(\cdot)$ as follows:

$$\int_T \phi(t) d\mu(t) = \left\{ \int_T x(t) d\mu(t) : x(\cdot) \in S^1_\phi \right\}.$$

We will denote the above integral as $\int \phi(\cdot)$, and call it the *Aumann integral.* We now state a result which will play a crucial role in the sequel. This is *Diestel's Theorem* [Diestel (1977)], which says that if $K: T \to 2^Y$ (here T is a finite measure space and Y is a separable Banach space) is an integrably bounded, convex, nonempty weakly compact valued correspondence, then S^1_K is weakly compact in $L_1(\mu, Y)$.

3. The Main Theorem

3.1 The Model. We now turn to the main result of the paper, i.e., the existence of a competitive equilibrium in economies with infinitely many commodities and agents.

Denote by E the *commodity space*, where E is an ordered separable Banach space whose positive cone E_+ has an *interior point* u. An *economy* $\mathcal{E}$ is a quadruple $[(T,\tau,\mu),X,\succsim,e]$ where

(1) (T,τ,μ) is a measure *space of agents*;
(2) $X : T \to 2^{E_+}$ is the *consumption correspondence*,
(3) $\succsim_t \subset X(t) \times X(t)$ is the *preference relation* of agent t,
(4) $e : T \to E_+$ is the *initial endowment* where for all $t \in T$, $e(t) \in X(t)$ and for all $t \in T$, $e(t)$ belongs to a norm compact subset of $X(t)$.

Denote the *budget set* of agent t at prices p by $B(t,p) = \{x \in X(t) : p \cdot x \leq p \cdot e(t)\}$. The *demand set* of agent t at prices p is defined as $D(t,p) = \{x \in B(t,p) : \text{for all } y \in B(t,p), x \succsim_t y\}$.

A *competitive equilibrium* for $\mathcal{E}$ is a price-consumption pair (p,f), $p \in E_+^*/\{0\}$, $f \in L_1(\mu, E_+)$ such that:

(i) $f(t) \in D(t,p)$ for almost all t in T, and
(ii) $\int_T f(t)d\mu(t) \leq \int_T e(t)d\mu(t)$.

3.2 Assumptions. The following assumptions which are standard in equilibrium analysis will be needed to prove our Main Theorem.

(3.1) (T,τ,μ) is a complete finite measure space.
(3.2) The correspondence $X : T \to 2^{E_+}$ is integrably bounded, closed, convex, nonempty, weakly compact valued, and it has a measurable graph, i.e., $G_X \in \tau \otimes \beta(E_+)$.
(3.2′) The correspondence $X : T \to 2^{E_+}$ is closed, convex, nonempty, norm compact valued and it has a measurable graph.
(3.3) (a) For each $t \in T$ and each $x \in X(t)$ the set $R(t,x) = \{y \in X(t) : y \succsim_t x\}$ is convex, and norm closed and the set $R^{-1}(t,x) = \{y \in X(t) : x \succsim_y\}$ is norm closed, (b) $\succsim_t$ is measurable in the sense that the set $\{(t,x,y) \in T \times E_+ \times E_+ : y \succsim_t x\}$ belongs to $\tau \otimes \beta(E_+) \otimes \beta(E_+)$.
(3.4) For all $t \in T$, there exists $z(t) \in X(t)$ such that $e(t) - z(t)$ belongs to the norm interior of E_+.

3.3 The Main Result. We are now ready to state our main result:

Main Theorem. *Let $\mathcal{E}$ be an economy satisfying (3.1)–(3.4). Then a competitive equilibrium exists in $\mathcal{E}$.*

A couple of comments are in order. Note that at a first glance, assumption (3.2) seems quite strong. In particular, traditionally the consumption sets are bounded from below *only*. However, in economies with a continuum of agents and commodities it has been shown by Zame (1987) that without the upper bound on the consumption sets, an equilibrium may not exist. Hence, if positive results need to be obtained the bound on the consumption sets must be imposed. Of course, once the bound on the consumption sets is imposed we are automatically in a world of either weakly compact or weak* compact consumption sets. For instance if the commodity space in any ordered (reflexive) Banach space and the consumption sets are norm bounded and (weakly) weak* closed, we can directly conclude by virtue of Alaoglu's Theorem [see Dunford-Schwartz (1966)] that the consumption sets are (weakly) weak* compact.

The weak compactness of consumption sets is needed to ensure that the set of all feasible allocations, i.e., $F = \{x \in S^1_X : \int_T x(t)d\mu(t) \leq \int_T e(t)d\mu(t)\}$ is weakly compact. In particular, under assumption (3.2) it follows from Diestel's Theorem that S^1_X is weakly compact and from this we can conclude that F is weakly compact as well. Notice that in economies with finitely (or even countably) many agents and infinitely many commodities the set of feasible allocations belongs to an order interval. Since order intervals are typically compact in the "compatible" topology that the commodity space is endowed with, the set of feasible allocations is always compact in the "compatible" topology. For instance if E is an ordered (reflexive) Banach space endowed with the (weak) weak* topology, one can easily see that order intervals are norm bounded and (weakly) weak* closed, hence, by Alaoglu's Theorem (weakly) weak* compact.

Since with a continuum of agents F does not belong to an order interval such an argument cannot be followed. However, one can replace assumption (3.2) by the fact that the set of all feasible allocations, i.e., F, is weakly compact. The proof of the Main Theorem remains unchanged in this case.

It is worth noting that even with a finite dimensional commodity space and a continuum of agents the set of all feasible allocations F is not compact in any topology. Nevertheless the use of the Fatou Lemma in several dimensions enables one to dispense with the bound on the consumption sets [see for instance Aumann (1966) or Schmeidler (1969)]. However, since Fatou's Lemma fails in infinite dimensional spaces [see for instance Rustichini (1989) or Yannelis (1990a)] a similar argument with that of Aumann or Schmeidler cannot be adopted. At this point we should mention that the coalitional approach adopted by Zame does not require the bound or the consumption set. In particular in this approach each allocation is always in an order internal which is compact typically in the topology that the commodity space is endowed with. However, as it was noted by Zame (1987) the existence of a competitive equilibrium for the conditional approach does not imply the existence of a competitive equilibrium for the Aumann individualistic approach adopted in this paper, *unless* the consumption sets are bounded. A more elaborate discussion of the connection of the two approaches can be found in Zame.

We now briefly discuss the assumption of convexity of preferences. One may wonder why the convexity assumption on preferences is needed. In particular, one of the nice features of the Aumann economy is that one can dispense with the assumption of convexity of preferences. In fact as Aumann (1966) showed, the Lyapunov Theorem will enable us to convexify the aggregate demand set and this makes applicable the standard fixed point argument. However, in infinite dimensional spaces Lyapunov's Theorem fails [see Diestel-Uhl (1977)] and consequently without convexity of preferences the aggregate demand set need not be convex. Hence, again if positive results need to be obtained the assumption of convexity of preferences must be imposed. [For further remarks on this issue see Rustichini-Yannelis (1990).]

4. An Auxiliary Theorem

As in Aumann (1966) in order to prove our Main Theorem, we first establish an auxiliary result. Recall that Aumann compactifies the economy and he proves a result for compact consumption sets [a similar auxiliary result was proved by Schmeidler (1969), as well]. Then using his auxiliary result (which is indeed the heart of the proof) he is able to complete the proof of his main theorem. A similar idea will be adopted here. In particular, we first establish an Auxiliary Theorem where consumption sets are norm compact. Once this result is available we proceed to complete the proof of the Main Theorem as follows. We construct a suitable family of truncated subeconomies each of which satisfies the assumptions of the Auxiliary Theorem. By appealing to the Auxiliary Theorem we can conclude that a competitive equilibrium exists in each subeconomy. Therefore, we obtain a net of equilibrium consumption-price pairs for the truncated subeconomies. The proof then is completed by extracting converging subnets whose limit is a competitive equilibrium for the original economy.

Below we state our Auxiliary Theorem which may be seen as the infinite dimensional extension of Aumann's (1966) Auxiliary Theorem.

Auxiliary Theorem. *Let $\mathcal{E}$ be an economy satisfying (3.1), (3.2′)–(3.4). Then a competitive equilibrium exists in $\mathcal{E}$.*

5. Proof of the Auxiliary Theorem

We begin by stating the following generalization of the Gale-Nikaido-Debreu Lemma proved in Yannelis (1985).

Main Lemma. *Let Y be a Hausdorff locally convex linear topological space whose positive cone Y_+ has an interior point u. Let $\Delta = \{p \in Y_+^* : p \cdot u = 1\}$. Suppose that the correspondence $\zeta : \Delta \to 2^Y$ satisfies the following conditions:*

(i) For all $p \in \Delta$ there exists $z \in \zeta(p)$ such that $p \cdot z \leq 0$,
(ii) $\zeta : \Delta \to 2^Y$ is weak u.s.c., (i.e., $\zeta : (\Delta, w^*) \to 2^Y$ is u.s.c.),*
(iii) for all $p \in \Delta$, $\zeta(p)$ is nonempty, convex and compact.

Then there exists $\bar{p} \in \Delta$, such that $\zeta(\bar{p}) \cap (-Y_+) \neq \phi$.

The Theorem below will be of fundamental importance for the proof of our equilibrium existence theorem. It should also be noted that results of the same nature with the Theorem below have found applications to equilibrium points of non-cooperative models of competition [see for instance Schmeidler (1973), Khan (1986), Khan-Papageorgiou (1987), and Yannelis (1987, 1990a)].

Theorem 5.1. *Let (T, τ, μ) be a complete, finite, separable measure space, Y be a separable Banach space, P be a metric space and $X : T \to 2^Y$ be an integrably bounded, convex, weakly compact, nonempty valued correspondence. Let $D : T \times P \to 2^Y$ be a nonempty, norm closed, convex valued correspondence such that:*

(i) for all $(t,p) \in T \times P$, $D(t,p) \subset X(t)$,
(ii) for each fixed $t \in T$, $D(t,\cdot)$ is norm u.s.c., and
(iii) for each fixed $p \in P$, $D(\cdot\,,p)$ has a measurable graph.

Then the correspondence $\phi : P \to 2^X$ defined by $\phi(p) = \{x \in S^1_X : x(t) \in D(t,p)$ for almost all $t \in T\}$ is nonempty valued and weakly u.s.c.

Proof. (a) Since for each fixed $p \in P$, $D(\cdot\,,p)$ has a measurable graph and it is nonempty valued, it follows from the Aumann measurable selection theorem that there exists a measurable function $f : T \to Y$ such that $f(t) \in D(t,p)$ for almost all t in T. Since $D(\cdot\,,\cdot)$ is integrably bounded, f is integrable and therefore $f \in \phi(p)$ for each $p \in P$. Hence, ϕ is nonempty valued.

(b) We now show that ϕ is weakly u.s.c. First, notice that by virtue of Diestel's Theorem, S^1_X is compact in the weak topology. Since by assumption (T, τ, μ) is a separable measure space, $L_1(\mu, Y)$ is separable [Dunford-Schwartz (1958, p. 381)]. Since S^1_X is a weakly compact subset of the separable Banach space $L_1(\mu, Y)$, by Theorem V.6.3 in Dunford-Schwartz (1958, p. 334), S^1_X is metrizable. Given that S^1_X with the weak topology is a compact metrizable space, in order to prove that ϕ is weakly u.s.c., it suffices to show that G_ϕ is closed in $P \times S^1_X$, where S^1_X is endowed with the weak topology. To this end let p_n, $(n = 1, 2, \ldots)$ be a sequence in P converging to p (in the metric topology), let y_n, $(n = 1, 2, \ldots)$ be a sequence in S^1_X converging weakly to y, and let $y_n \in \phi(p_n)$. We must show that $y \in \phi(p)$. Let $A_i = \operatorname{con} \bigcup_{j>i} y_j$ for $i = 1, 2, \ldots$. Since y_n converges weakly to y by Mazur's Theorem [see for instance Dunford-Schwartz (1958), Corollary 14, p. 422)] for each $i = 1, 2, \ldots$, there exists a sequence $\{z^i_n\}$ in A_i converging in norm to y. For any $\delta > 0$, we can find n_1 such that $\|z^1_{n_1} - y\| < \delta$. Similarly for $m > 1$, we can find n_m such that $\|z_{n_m} - y\| < \frac{\delta}{m}$. Continuing this process we can construct a sequence, appropriately relabeled $\{z_n\}$, such that $z_n \in \operatorname{con} \bigcup_{j \geq n} y_j$, and z_n converges in norm to y. Without loss of generality we can assume that

$z_n(t)$ converges in norm to $y(t)$ (otherwise pass to a subsequence) for all $t \in T/S$, where $S \subset T$, $\mu(S) = 0$. Fix t in T/S. Since for each fixed $t \in T$, $D(t, \cdot)$ is norm for u.s.c., for every $\delta > 0$ we can find n, $(n = 1, 2, \ldots)$ such that for all $n_0 \geq n$ we have $D(t, p_{n_0}) \subset D(t,p) + \delta B$, (where B is the open unit ball in Y). Hence

$$\text{con} \bigcup_{n_0 \geq n} D(t, p_{n_0}) \subset D(t,p) + \delta B \Rightarrow z_n(t) \in D(t,p) + \delta B$$

$$\Rightarrow y(t) \in D(t,p) + \delta B.$$

By letting δ go to zero, we conclude that $y(t) \in D(t,p)$. Since $t \in T/S$ was arbitrary, $y(t) \in D(t,p)$ for almost all t in T. Finally since $D(t,p) \subset X(t)$ for all $t \in T$ and $D(\cdot, \cdot)$ is integrably bounded, we can conclude that $y \in \phi(p)$. This completes the proof of the Theorem.

Corollary 5.1. *Let $D : T \times P \to 2^Y$ be a correspondence satisfying all the assumptions of Theorem 5.1. Then,*

(i) $\int D(\cdot, p)$ is nonempty, and
(ii) $\int D(t, \cdot)$ is weakly u.s.c.

Proof. (i) Since for each $p \in P$, $\psi(p) \neq \phi$, it follows that $\int D(\cdot, p)$ is nonempty.

(ii) We now show that $\int D(t, \cdot)$ is weakly u.s.c. Define $\psi : P \to 2^{S^1_X}$ by $\psi(p) = \{y \in S^1_X : y(t) \in D(t,p)$ for almost all t in $T\}$. Let $f : S^1_X \to Y$ be a mapping defined by $f(\psi(p)) = \int D(t,p)$. Clearly f is norm continuous and linear. It is a standard result [see for instance Aliprantis-Burkinshaw (1985, Theorem 9.16, p. 139) or Dunford-Schwartz (1958, Theorem 15, p. 422)] that f is also weakly continuous. By Theorem 5.1, ψ is weakly u.s.c. and so is $f(\psi)$. Hence, $\int D(t, \cdot)$ is weakly u.s.c. and this completes the proof of the Corollary.

Remark 5.1. The separability assumption on the measure space (T, τ, μ) in Theorem 5.1 and Corollary 5.1 is not needed provided the reader follows the argument in Yannelis (1990a, Theorem 5.4 and Remark 5.1).

Observe that conclusions (i) and (ii) of the above corollary have been proved by Aumann (1965, 1976) for $Y = \mathbb{R}^\ell$. Hence the above corollary may be seen as an extension of Aumann's result. Conclusion (ii) of

Corollary 5.1 can be also obtained as a corollary of the infinite dimensional version of Fatou's Lemma proven in Yannelis (1988). For a further discussion on this see Yannelis (1990a).

The following result proved in Yannelis (1988, Lemma 3.1) is the infinite dimensional extension of Theorem 4 in Aumann (1965).

Lemma 5.1. *Let (T, τ, μ) be a complete finite measure space and Y be a separable Banach space. Let $\phi : T \to 2^Y$ be a closed, convex valued correspondence such that $\phi(t) \subset X(t)$ for all $t \in T$, where $X : T \to 2^Y$ is an integrably bounded, nonempty, weakly compact, convex valued correspondence. Then,*

$$\int_T \phi(t) d\mu(t) \quad \textit{is weakly compact.}$$

Notice that Aumann (1965) does not require $\phi(\cdot)$ to be convex valued. However, it can be easily shown that the above result is false without the convex valuedness of ϕ [see Rustichini (1989) or Yannelis (1990a)].

We now state a recent result proved in Khan-Vohra (1985, Theorem B, p. 331).

Lemma 5.2. *Let $\{z_k : k \in K\}$ be a net in B, where B is a weakly compact subset of a Banach space, and suppose that z_k converges weakly to z. Then we can extract a sequence $\{z_n : n = 1, 2, \ldots\}$ from the net $\{z_k : k \in K\}$ which converges weakly to z.*

With all these preliminary results out of the way, we can now complete the proof of the Auxiliary Theorem.

Let $\Delta = \{p \in E_+^* : p \cdot u = 1\}$ be the price space. It follows from Alaoglu's Theorem [Jameson (1970, p. 123)] that Δ is weak* compact. Moreover, since E is a separable Banach space, Δ is metrizable, [Dunford-Schwartz (1958, p. 426)]. For $p \in \Delta$ and $t \in T$, let the budget set be $B(t,p) = \{x \in X(t) : p \cdot x \leq p \cdot e(t)\}$. Since for each $t \in T$, $X(t)$ is norm compact and Δ is weak* compact, the bilinear form $(p, x) \to p \cdot x$ is jointly continuous [see for instance Yannelis-Zame (1986) Lemma A, p. 107]. Hence, it follows from assumption (3.4) that for each fixed $t \in T$, $B(t, \cdot)$ is continuous and a standard argument can be adopted to show that for each fixed $t \in T$, $D(t, \cdot)$ is u.s.c. in the sense that the set $\{p \in \Delta : D(t,p) \subset V\}$ is weak* open in Δ for every norm space subset V of E_+.

Since $\succsim_t$ is convex, transitive and complete, a standard argument shows that $D(\cdot\,,\cdot)$ is convex and nonempty valued. We will show that for each fixed $p \in \Delta$, $B(\cdot\,,p)$ has a measurable graph. To see this for $p \in \Delta$, define $g_p : T \times E \to [-\infty, \infty]$ by $g_p(t,x) = p \cdot x - p \cdot e(t)$. Clearly, g_p is measurable in t and continuous in x, and hence by a standard result [see for instance Yannelis (1990, Proposition 3.1)], $g_p(\cdot\,,\cdot)$ is jointly measurable. Therefore, $g_p^{-1}([-\infty,0]) \in \tau \otimes \beta(E)$. It can be easily checked that

$$\begin{aligned} G_{B(\cdot\,,p)} &= \{(t,x) \in T \times X(t) : p \cdot x \leq p \cdot e(t)\} \\ &= g_p^{-1}([-\infty,0]) \cap G_X. \end{aligned}$$

Since by assumption $X(\cdot)$ has a measurable graph it follows that for each fixed $p \in \Delta$, $G_{B(\cdot\,,p)} \in \tau \otimes \beta(E)$. Since (T,τ,μ) is a complete measure space and $B(\cdot\,,\cdot)$ is closed valued, it follows [see for instance Yannelis (1990, Lemma 3.1)] that for each fixed $p \in \Delta$, $B(\cdot\,,p)$ is lower measurable. Hence, by Castaing's Representation Theorem [see Yannelis (1990)] there exists a family $\{f_n : n = 1,2,\dots\}$ of measurable functions $f_n : T \to E$ such that for all $t \in T$, $\mathrm{cl}\,\phi\{f_n(t) : t \in T\} = B(t,p)$ (where cl denotes norm closure). For $n = 1,2,\dots$ let

$$D_n(t,p) = \{y \in B(t,p) : y \succsim_t f_n(t)\}.$$

Since $\succsim_t$ and $B(\cdot\,,p)$ have measurable graphs so does $D_n(\cdot\,,p)$. We wish to show that $D(t,p) = \bigcap_{n=1}^{\infty} D_n(t,p)$. Obviously, $D(t,p) \subset D_n(t,p)$ for each n, $(n = 1,2,\dots)$. We now show that $\bigcap_{n=1}^{\infty} D_n(t,p) \subset D(t,p)$. Suppose otherwise, i.e., there exists $z \in \bigcap_{n=1}^{\infty} D_n(t,p)$ and $z \notin D(t,p)$, i.e., there exists $y \in B(t,p)$ such that $y \succsim_t x$. Notice that by assumption the set $\{w \in B(t,p) : w \succsim_t z\}$ is norm closed in $B(t,p)$. Since the family $\{f_n(t) : n = 1,2,\dots\}$ is norm dense in $B(t,p)$ we can find an n_0 such that $f_{n_0}(t) \succsim_t z$, a contradiction. Hence, $D(t,p) = \bigcap_{n=1}^{\infty} D_n(t,p)$ and since for each fixed $p \in \Delta$, $D_n(\cdot\,,p)$ $(n = 1,2,\dots)$ have measurable graph so does $D(\cdot\,,p)$.

Define the excess demand correspondence $\zeta : \Delta \to 2^E$ for the economy $\mathcal{E}$ by

$$\zeta(p) = \int_T D(t,p)\,d\mu(t) - \int_T e(t)\,d\mu(t).$$

We must show that ζ satisfies all the conditions of the Main Lemma. Clearly, for each $p \in \Delta$, $\zeta(p)$ convex valued and $p \cdot \zeta(p) \leq 0$. Since for each fixed $p \in \Delta$, $D(\cdot\,,p)$ has a measurable graph and is nonempty valued, the first conclusion of Corollary 4.1 assures that $\int D(\cdot\,,p)$ is nonempty and therefore, $\zeta(p)$ is nonempty for each $p \in \Delta$. Since for each fixed $t \in T$, $D(t,\cdot)$ is u.s.c. and $D(\cdot\,,\cdot)$ is convex, closed valued and $D(t,p) \subset X(t)$ for all $t \in T$, where $X : T \to 2^{E_+}$ is integrably bounded, norm compact, convex, nonempty valued, it follows from the second conclusion of Corollary 5.1 that $\int D(t,\cdot)$ is weakly u.s.c. and so ζ is weakly u.s.c. as well. Finally, it follows from Lemma 5.1 that $\int_T D(t,p)d\mu(t)$ is weakly compact, and hence, $\zeta(p)$ is weakly compact for each $p \in \Delta$. Consequently, ζ satisfies all the assumptions of the Main Lemma and therefore that there exist $(\bar{p},\bar{x})$ such that $\bar{x} \in \zeta(\bar{p})$ and $\bar{x} \leq 0$, i.e., $\bar{x} = \int_T f(t)d\mu(t) - \int_T e(t)d\mu(t) \leq 0$ and $f(t) \in D(t,p)$ for almost all $t \in T$. Hence $(\bar{p}, f)$ is a competitive equilibrium and this completes the proof of the Auxiliary Theorem.

6. Proof of the Main Theorem

Let $\mathcal{F}$ be a family of all nonempty, norm compact, convex subsets of E_+ containing the initial endowments. For each $F \in \mathcal{F}$ define the consumption correspondence $X^F : T \to 2^{E_+}$ by

$$X^F = F \cap X(t).$$

Moreover, for each $F \in \mathcal{F}$ let $\succsim_t^F$ be the preference relation of agent t induced on F. Let $S^1_{X^F} = \{x \in L_1(\mu, E_+) : x(t) \in X^F(t)$ for almost all t in $T\}$.

We now have a truncated economy $\mathcal{E}^F = [(T,\tau,\mu), X^F, \succsim^F, e]$ which is easily seen that satisfies all the assumptions of the Auxiliary Theorem. Consequently, a competitive equilibrium in $\mathcal{E}^F$ exists, i.e., there exist (p_F, x_F), $p_F \in E^*_+/\{0\}$, $x_F \in S^1_X F$ such that:

(i) $x_F(t) \in D(t,p_F)$ for almost all t in T, and

(ii) $\int_T x_F(t)d\mu(t) \leq \int_T e(t)d\mu(t)$.

Since for each X is weakly compact, nonempty, and convex valued, by Diestel's Theorem, S^1_X is weakly compact in $L_1(\mu, E_+)$. Observe that for each $F \in \mathcal{F}$, $x_F \in A = \{y \in S^1_X : \int_T y(t)d\mu(t) \leq \int_T e(t)d\mu(t)\}$. It can be easily checked that A is convex and norm closed and as a consequence of the Separation Theorem weakly closed. Since S_X^1 is weakly compact, we can conclude that A is weakly compact as well. Notice that for each $F \in \mathcal{F}$, $x_F \in A$, and $\mathcal{F}$ is net directed by inclusion. Hence, the net $\{x_F : F \in \mathcal{F}\}$ has a subset, still denoted by x_F which converges weakly to $\bar{x} \in A$. Moreover, for each $F \in \mathcal{F}$, p_F lies in $\Delta = \{q \in Y_+^* : q \cdot u = 1\}$ and the latter set is weak* compact. Hence, from the equilibrium net $\{(p_F, x_F) : F \in \mathcal{F}\}$ we can always extract convergent subnets. It is clear that $\int_T \bar{x}(t)d\mu(t) \leq \int_T e(t)d\mu(t)$, i.e., $\bar{x}$ is a feasible allocation. We must now show that the limiting allocation $\bar{x}(t)$ is maximal in the budget set, for almost all t in T, in order to complete the proof. We know that for each $F \in \mathcal{F}$, $x_F(t) \in D(t, p_F)$ for almost all t in T, and x_F converges weakly to $\bar{x}$. Since the net $\{x_F : F \in \mathcal{F}\}$ lies in the weakly compact set A, by Lemma 5.2 we can extract a sequence x_n, $(n = 1, 2, \ldots)$ from the net $\{x_F : F \in \mathcal{F}\}$ which converges weakly to $\bar{x} \in A$. Corresponding to the sequence x_n, $(n = 1, 2, \ldots)$ we can also extract a sequence p_n, $(n = 1, 2, \ldots)$ from the net $\{p_F : F \in \mathcal{F}\}$. Obviously, p_n belongs to Δ, and p_n has a subsequence still denoted by p_n which converges weak* to $\bar{p}$.

Therefore, we have a sequence $\{(p_n, x_n) : n = 1, 2, \ldots\}$ such that p_n converges weak* to $\bar{p}$ and x_n converges weakly to $\bar{x}$. Since $x_n(t) \in D(t, p_n)$ for almost all $t \in T$ and $D(t, p_n)$ is contained in $X(t)$ we have that $x_n \in \phi(p_n) = \{y \in S_X^1 : y(t) \in D(t, p_n)$ for almost all t in $T\}$. By Theorem 5.1 the correspondence $\phi : \Delta \to 2^{S_X^1}$ is weakly u.s.c. and closed valued and thus we can conclude that $\bar{x} \in \phi(\bar{p})$, i.e., $\bar{x}(t) \in D(t, \bar{p})$ for almost all t in T. Hence, $\bar{x}(t)$ is maximal in the budget set for almost all t in T and this completes the proof of the Theorem.

7. Concluding Remarks

Remark 7.1. As in Aumann (1965), we assumed that agents' preferences are complete. Schmeidler (1969) showed that the completeness assumption can be dropped from the Aumann model. However, this is not the case with infinitely many commodities. Specifically, without the completeness assumption on preferences, Mas-Colell (1974) showed that even if preferences are convex, the demand set may not be convex and therefore the aggregate demand set may not be convex. Of course such a problem does not arise in Schmeidler's framework since with an atomless measure space of agents and finitely many commodities the aggregate demand set is always convex as a consequence of the Lyapunov Theorem. [This is also the case in Rustichini-Yannelis (1990) where the economy has "many more" agents than commodities and there is a convexifying effect on aggregation.]

Remark 7.2. If the convexity assumptions on preferences is relaxed from our model, once we assume that the measure space of agents is atomless, then we can easily prove the existence of an approximate or δ-competitive equilibrium. In particular, one can convexify the demand set $D(t,p)$ by taking its closed convex hull, i.e., $\overline{\text{con}}\, D(t,p)$. [Notice that for each fixed $t \in T$, $\overline{\text{con}}\, D(t,p)$ is u.s.c. and for each fixed $p \in \Delta$, $\overline{\text{con}}\, D(\cdot\,,p)$ has a measurable graph.] Note that by Theorem 1 in Khan (1986) [see also Yannelis (1990a, Theorem 6.3)] we have that

$$\text{cl} \int_T D(t,p)d\mu(t) = \int_T \overline{\text{con}}\, D(t,p)d\mu(t).$$

Proceeding now as in the proof of the Auxiliary Theorem one can easily show that $x(t) \in D(t,p)$ for almost all t in T and $\| \int_T x(t)d\mu(t) - \int_T e(t)d\mu(t)\| < \delta$. Note that now the completeness assumption on preferences is not needed (recall Remark 7.1).

Remark 7.3. The space $C(X)$, i.e., the space of continuous functions on the compact metric space X with the sup norm is an ordered separable Banach space whose positive cone has a nonempty norm interior. Hence, the Main Theorem covers $C(X)$. It is important to note that in this space even if the set of agents is finite one cannot relax the bound from the consumption sets. In particular, since order intervals are not compact in any topology, one cannot conclude that the set of all feasible allocations (which always lie in an order interval) is compact. Finally, it is important to note that our Main Theorem covers $L_\infty(\Omega)$, i.e., the space of essentially bounded measurable functions on the measure space Ω, with the sup norm. This is due to the fact that weakly compact subsets of L_∞ are norm separable [see for instance Diestel-Uhl (1977, Theorem 13, p. 252)].

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