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Efficiency and incentive compatibility in differential information economies★

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Summary. We introduce several efficiency notions depending on what kind of expected utility is used (ex ante, interim, ex post) and on how agents share their private information, *i.e.,* whether they redistribute their initial endowments based on their own private information, or common knowledge information, or pooled information. Moreover, we introduce several Bayesian incentive compatibility notions and identify several efficiency concepts which maintain (coalitional) Bayesian incentive compatibility.

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1 Introduction

An exchange economy with differential information consists of a finite set of agents, each of whom is characterized by a random (state dependent) utility function, random initial endowment, a private information set, and a prior. In such an economy the definition of efficiency (or Pareto optimality) is not immediate as was first alluded to in seminal papers by Wilson (1978) and by Myerson (1979). (The latter considered the Harsanyi framework rather than an exchange economy with differential information.) In particular, two main problems arise. First, if we assume that agents make agreements (contracts) before the state of nature is realized, it is important to know what kind of expected utility we adopt, *i.e.,* ex ante or interim. Moreover, how does the choice of the expected utility change the outcome? Secondly and most importantly, when all agents make a redistribution of their initial endowments, what kind of information do they use? That is, do they pool their informa-

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tion, or do they use common knowledge information, or do they just make redistribution based on their own information?

Before we proceed, it may be useful to accept the fact that there is no single definition of efficiency which universally works for all environments. In fact, since the information is decentralized in differential information economies, the incentive problem becomes a critical issue for a mechanism which allocates resources according to the reports of agents. This problem was first raised by Myerson (1979) and Holmström-Myerson (1983) in the Harsanyi framework. The key point is that an efficient allocation may not be Bayesian incentive compatible, *i.e.,* the set of efficient allocations is much larger than the set of Bayesian incentive compatible allocations. Our main purpose is to focus on notions of efficiency which are incentive compatible.

Several of the interim efficiency concepts that we introduce in this paper are stronger than those of Holmström-Myerson (1983), but we think they are the proper concepts to capture the efficiency idea in differential information economies. The main assumption we impose is that the net trades are private information measurable.[1] If such a condition is not satisfied, *i.e.,* a proposed net trade is not measurable with respect to private information, then it may create incentive problems and contracts may not be viable (see Example 3.1 as well as Example 6.1). Consequently, it is reasonable to impose the private information measurability condition on allocations.[2]

With the private measurability assumption, every feasible allocation turns out to be Bayesian incentive compatible. Indeed, no single agent can lie and become better off simply because if he/she becomes better off by lying, at least one other agent should be worse off by feasibility, which is impossible by the private information measurability.[3] This weak property of Bayesian incentive compatibility suggests that a stronger Bayesian incentive compatibility notion, *coalitional Bayesian incentive compatibility* may be appropriate. The idea is that no coalition can become better off by reporting false events. That is, in terms of game theory, truth-telling is a coalitional (or strong) Nash equilibrium when agents are asked to report their private information events.

Based on the "proper" efficiency notion and the private information measurability condition, we show that any "proper" efficient allocation is coalitionally Bayesian incentive compatible. This means that if we adopt

[1] The endowment is an initial signal of states and every agent has a private information generated by this signal. This means that the private information measurability of net trades is equivalent to that of allocations.

[2] It has been shown in Krasa-Yannelis (1994) that private measurability is a necessary and sufficient condition for coalitional Bayesian in entire compatibility in the one good per state differential information economy.

[3] All agents except the single liar do not distinguish the true state and the false state. The private information measurability assumption implies that their allocations in the false state are the same as in the true state. Since lying does not change the total endowment in the true state, there is no way for the single liar to become better off.

certain efficiency concepts, the incentive issue (individual or coalitional) need not be considered. It should be noted that a Holmström-Myerson type efficiency notion with the private information measurability condition does not have this property.

Finally, we consider an (interim) efficiency notion without the individual measurability assumption and propose a notion of incentive efficiency. This concept corresponds to the interim efficiency concept of Myerson (1979) and Holmström-Myerson (1983) that they have introduced for the Harsanyi framework and it is different from our other interim efficiency concepts. As in Myerson (1979), it is shown to exist whenever the utility functions are affine. This argument favors our earlier concepts of interim efficiency which exist assuming only concavity of the utility functions.

The paper is organized as follows: Section 2 outlines the basic mathematical notation and definitions. The description of the differential information economy is given in Section 3. We propose several concepts of incentive compatibility in Section 4. In Section 5, we define efficiency concepts in differential information and characterize their properties. The relationship between efficiency and incentive compatibility is examined in Section 6. There are some remarks on individual rationality in Section 7. In section 8, We show the existence of individually rational and efficient allocation. Without measurability, incentive efficiency is defined and analyzed in Section 9.

2 Notation and definitions

We begin with some notation and definitions.

2.1 Notation

$|A|$ denotes the number of elements in the set A.
2^A denotes the family of all subsets of A.
$\setminus$ denotes the set theoretic subtraction.
If A is a set, we denote by χ_A the characteristic function having the property that $\chi_A(\omega)$ is one if $\omega \in A$ and it is zero otherwise.

2.2 Definitions

Let $(\Omega, \mathscr{F}, \mu)$ be finite measure space, and X be a Banach space. Following Diestel-Uhl (1977), the function $f : \Omega \to X$ is called *simple* if there exist $x_1, x_2, \ldots, x_n$ in X and $A_1, A_2, \ldots, A_n$ in $\mathscr{F}$ such that $f = \sum_{i=1}^{n} x_i \chi_{A_i}$. A function $f : \Omega \to X$ is said to be *μ-measurable* if there exits a sequence of simple functions $f_n : \Omega \to X$ such that $lim_{n\to\infty} \|f_n(\omega) - f(\omega)\| = 0$ for almost all $\omega \in \Omega$. A μ-measurable function $f : \Omega \to X$ is *Bochner integrable* if there exists a sequence of simple functions $\{f_n : n = 1, 2, \ldots\}$ such that

$$\lim_{n\to\infty} \int_{\Omega} \|f_n(\omega) - f(\omega)\| d\mu(\omega) = 0.$$

In this case, we define for each $A \in \mathscr{F}$, the *integral* to be

$$\int_A f(\omega)d\mu(\omega) = \lim_{n\to\infty}\int_A f_n(\omega)d\mu(\omega).$$

It can be shown [see Diestel-Uhl (1977), Theorem 2, p.45] that if $f:\Omega\to X$ is a μ-measurable function, then f is Bochner integrable if and only if $\int_\Omega \|f(\omega)\|d\mu(\omega)<\infty$. It is important to note that the *Dominated Convergence Theorem* holds for Bochner integrable functions. In particular, if $\{f_n:\Omega\to X: n=1,2,\ldots\}$ is a sequence of Bochner integrable functions such that $\lim_{n\to\infty} f_n(\omega)=f(\omega)\mu\text{-}a.e.$, and $\|f_n(\omega)\|\le g(\omega)\mu\text{-}a.e.$, where $g:\Omega\to \boldsymbol{R}$ is an integrable function, then f is Bochner integrable and $\lim\int_\Omega \|f_n(\omega)-f(\omega)\|d\mu(\omega)=0$ [see Diestel and Uhl (1977), Theorem 3, p.45].

Denote by $L_p(\mu,X)$ with $1\le p<\infty$ the space of equivalence classes of X-valued Bochner integrable functions $x:\Omega\to X$ normed by

$$\|x\|_p = \left[\int_\Omega \|x(\omega)\|^p d\mu(\omega)\right]^{\frac{1}{p}} < \infty.$$

It is a standard result that normed by the functional $\|\cdot\|_p$ above, $L_p(\mu,X)$ becomes a Banach space [see Diestel-Uhl (1977), p.50].

We will denote by $L_\infty(\mu,X)$ the space of equivalence classes of essentially bounded Bochner integrable functions $x:\Omega\to X$ normed by

$$\|x\|_\infty = \operatorname{ess\,sup}\|x\| = \inf\{\varepsilon\in\boldsymbol{R}_+ : \mu\{\omega\in\Omega : \|x(\omega)\|>\varepsilon\}=0\}.$$

Normed by the functional $\|\cdot\|_\infty$, $L_\infty(\mu,X)$ with $1\le p<\infty$ becomes a Banach space [Diestel-Uhl (1977, p.50)]. It is well-known that $L_q(\mu,X^*)$ is the dual of $L_p(\mu,X)$, where $1\le p\le\infty$ and $1/p+1/q=1$, and the value $\omega\cdot x$ of $x\in L_p(\mu,X)$ at $\omega\in L_q(\mu,X^*)$ is defined by

$$w\cdot x = \int_\Omega [w(\omega)\cdot x(\omega)]d\mu(\omega).$$

Recall that $\sigma(L_p(\mu,X),L_q(\mu,X^*))$ is defined as the weakest topology on $L_p(\mu,X)$ for which a net $x^\lambda\to x$ if and only if $w\cdot x^\lambda\to w\cdot x$ for all $w\in L_q(\mu,X^*)$. We call this topology as *weak topology* and the convergence as *weak convergence.* A function $f:X\to\boldsymbol{R}$ *is weakly upper semicontinuous* if $\lim\sup f(x^\lambda)\le f(x)$, *weakly lower semicontinuous* if $\lim\inf f(x^\lambda)\ge f(x)$, and *weakly continuous* if it is both weakly upper semicontinous and weakly lower semicontinuous, whenever $x^\lambda\to x$ weakly.

Now we state basic results on Banach lattices [see Aliprantis-Burkinshaw (1985) for details]. A Banach space X is a *Banach lattice* if there exists an ordering $\ge$ on X with the following properties:

(1) $x\ge y$ implies $x+z\ge y+z$ for every $z\in X$,
(2) $x\ge y$ implies $\lambda x\ge\lambda y$ for every $\lambda\in\boldsymbol{R}_+$,
(3) for all $x,y\in X$, there exist a supremum $x\vee y$ and an infimum $x\wedge y$,
(4) $|x|\ge|y|$ implies $\|x\|\ge\|y\|$ for every $x,y\in X$.

For $x, y \in X$, define the *order interval* $[x, y]$ by $[x, y] = \{z \in X : x \leq z \leq y\}$. Note that $[x, y]$ is convex and norm closed, hence weakly closed (Mazur's Theorem). Cartwright (1974) has shown that if X is a Banach lattice with order continuous norm or equivalently has weakly compact order intervals, then $L_p(\mu, X)$ with $1 \leq p < \infty$ has weakly compact order intervals, as well.[4]

All the results of the paper hold true for any Banach space $L_p(\mu, X), 1 \leq p \leq \infty$. However, we will restrict ourselves to $L_1(\mu, X)$.

3 Differential information economies

Below we define the notion of an economy with differential information (or Radner-type economy). Let $(\Omega, \mathscr{F}, \mu)$ be a probability measure space denoting the states of the world and Y be an ordered Banach space denoting the commodity space.[5] An *economy with differential information* is described by $\mathscr{E} = \{(X_i, u_i, \mathscr{F}_i, \mu, e_i) : i \in I\}$, where

(1) $X_i : \Omega \to 2^{Y+}$ is the *random consumption set correspondence of agent* $i \in I$.
(2) $u_i : \Omega \times Y_+ \to \boldsymbol{R}$ is the *random utility function of agent* $i \in I$.
(3) $\mathscr{F}_i$ is a (finite) measurable partition[6]of Ω denoting the *private information of agent* $i \in I$.[7]
(4) μ is a probability measure on Ω denoting the *common prior* of each agent.
(5) $e_i : \Omega \to Y_+$ is an $\mathscr{F}_i$-measurable and Bochner integrable function denoting the *random initial endowment* of agent $i \in I$, where $e_i(\omega) \in X_i(\omega) \mu$*-a.e.*

Let us denote by L_i the set of all $\mathscr{F}_i$-measurable and Bochner integrable functions from Ω to Y, *i.e.*, $L_i = \{x_i \in L_1(\mu, Y) : x_i$ is $\mathscr{F}_i$-measurable$\}$. Denote by L_{X_i}, the set of all $\mathscr{F}_i$-measurable and Bochner integrable selections from the correspondence X_i, *i.e.*, $L_{X_i}, = \{x_i \in L_1(\mu, Y) : x_i$ is $\mathscr{F}_i$-measurable and $x_i(\omega) \in X_i(\omega) \mu$*-a.e.*$\}$. Let $L = \prod_{i \in I} L_i$ and $L_X = \prod_{i \in I} L_{X_i}$. We assume that for each $i \in I$ and each $x_i \in Y_+, u_i(\cdot, x_i)$ is integrably bounded. Denote $\overline{e} = \sum_{i \in I} e_i$.

The *ex ante expected utility function* $\overline{V}_i : L_{X_i} \to \boldsymbol{R}$ of agent i is defined by

[4] $x_\lambda \downarrow 0$ means that x_λ is a decreasing net with $\inf x_\lambda = 0$. A Banach lattice X is said to have an *order continuous norm* if $x_\lambda \downarrow 0$ in X implies $\|x_\lambda\| \downarrow 0$. If X is a Banach lattice, X has an order continuous norm if and only if any order interval is weakly compact.

[5] It is important to note that even if we assume that our commodity space $Y = \boldsymbol{R}^\ell$ (where $\boldsymbol{R}^\ell$ is the ℓ-fold Cartesian product of the reals $\boldsymbol{R}$), the space $L_p(\mu, \boldsymbol{R}^\ell), 1 \leq p \leq \infty$ is still infinite dimensional (in view of the continuum of states). Hence, to assume that $Y = \boldsymbol{R}^\ell$ does not change in any way the arguments of the main results of the paper. As a matter of fact, even if we have just one good, i.e., $Y = \boldsymbol{R}$, we will need to work with $L_p(\mu, \boldsymbol{R}), 1 \leq p \leq \infty$ which is an infinite dimensional space.

[6] One may assume that $\mathscr{F}_i$ is a sub-σ-algebra of $\mathscr{F}$. The results of the paper remain unaffected.

[7] Throughout our analysis, we assume that information partitions $\{\mathscr{F}_i\}_{i \in I}$ are common knowledge in the sense of Aumann.

$$\overline{V}_i(x_i) = \int_{\Omega} u_i(\omega, x_i(\omega))d\mu(\omega).$$

We call a set of states an event. An event E_i, which is an element of $\mathscr{F}_i$, is the maximal set of states that agent i cannot distinguish. Let $E_i(\omega)$ denote the element of $\mathscr{F}_i$ which contains $\omega \in \Omega$. This means that when a true state ω occurs, agent i knows only that $E_i(\omega)$ occurs instead. Assume that $\mu(E_i(\omega)) > 0$ for every $i \in I$ and every $\omega \in \Omega$. The *interim (conditional) expected utility function* $V_i : \Omega \times L_{X_i} \to \boldsymbol{R}$ of agent i is defined by[8]

$$V_i(\omega \cdot x_i) = \frac{1}{\mu(E_i(\omega))} \int_{E_i(\omega)} u_i(\omega', x_i(\omega'))d\mu(\omega').$$

Lemma 3.1.1: If, for every $i \in I, u_i(\omega, \cdot)$ is continuous for each $\omega \in \Omega$, then

(1) $\overline{V}_i$ is continuous,
(2) $V_i(\omega, \cdot)$ is continuous for each $\omega \in \Omega$.

Proof: Since, for every $i \in I, u_i(\omega, \cdot)$ is continuous for every $\omega \in \Omega$ and $u_i(\cdot, x_i)$ is integrably bounded for every $x_i \in Y_+$, the result follows directly from the Dominated Convergence Theorem [see Diestel-Uhl (1977), Theorem 3, p.45]. □

Lemma 3.1.2: For every $i \in I$, if $u_i(\omega, \cdot)$ is upper semicontinuous and concave for every $\omega \in \Omega, \overline{V}_i$ is weakly upper semicontinous and concave and $V_i(\omega, \cdot)$ is weakly upper semicontinuous and concave for every $\omega \in \Omega$.

Proof: See Theorem 2.8 in Balder-Yannelis (1993). □

Lemma 3.1.3: For every $i \in I, u_i(\omega, \cdot)$ is continuous and affine for every $\omega \in \Omega$ if and only if $\overline{V}_i$ is weakly continuous and $V_i(\omega, \cdot)$ is weakly continuous for every $\omega \in \Omega$.

Proof: See Corollary 2.7 and Corollary 2.9 in Balder-Yannelis (1993). □

Lemma 3.1.4: For every $i \in I$, if u_i is $\mathscr{F}_i$-measurable, then it follows that

$$V_i(\omega,\ x_i) = u_i(\omega, x_i(\omega)).$$

[8] One could allow agents to have different priors as follows: Let $q_i : \Omega \to \boldsymbol{R}_{++}$ be the *prior* of agent i, which is a Radon-Nikodym derivative of μ having the property that $\int_{\Omega} q_i(\omega)d\mu(\omega) = 1$. Then the interim expected utility function $V_i : \Omega \times L_{X_i} \to \boldsymbol{R}$ of agent i is defined by

$$V_i(\omega, x_i) = \int_{E_i(\omega)} u_i(\omega', x_i(\omega'))q_i(\omega'|E_i(\omega))d\mu(\omega'),$$

where

$$q_i(\omega'|E_i(\omega)) = \begin{cases} q_i(\omega')/\int_{E_i(\omega)} q_i(s)d\mu(s) \text{if } \omega' \in E_i(\omega) \\ 0 \qquad \text{otherwise.} \end{cases}$$

The results of the paper will remain valid under the above interim expected utility framework, but we choose not to adopt it for simplicity and convenience.

Proof: For every $i \in I$, since x_i and u_i are $\mathscr{F}_i$-measurable, $u_i(\omega', x_i(\omega'))$ is constant on $E_i(\omega)$. Therefore,

$$\int_{E_i(\omega)} u_i(\omega', x_i(\omega'))d\mu(\omega') = \mu(E_i(\omega))u_i(\omega, x_i(\omega))$$

and the conclusion follows. □

The *set of feasible allocations* is given by $\boldsymbol{A} = \{x \in L_X : \sum_{i \in I} x_i = \sum_{i \in I} e_i\}$. For each i, an element $z_i \in L_i$ with $z_i = x_i - e_i$ is a *net trade of agent i*. The *set of feasible net trades* is given by $\boldsymbol{Z} = \{z \in L : \sum_{i \in I} z_i = 0\}$. Let $\hat{Z} = \{\hat{z} \in \prod_{i \in I} Y_i : \sum_{i \in I} \hat{z}_i = 0\}$, where $Y_i = Y$ for every $i \in I$. Notice that the initial endowment vector denoted by $e = (e_i)_{i \in I}$ is an element of L_X. Let $L^0_{X_i} = \{x_i \in L_1(\mu, Y_+) : x_i(\omega) \in X_i(\omega) \mu\text{-}a.e.\}$ and $L^0_X = \prod_{i \in I} L^0_{X_i}$. Define the *set of ex post allocations by* $\boldsymbol{A}^0 = \{x \in L^0_X : \sum_{i \in I} x_i = \sum_{i \in I} e_i\}$. *For each partition* $\mathscr{G}$ *of* Ω, *define* $\boldsymbol{A}(\mathscr{G}) = \{x \in L^0_X : x_i$ *is* $\mathscr{G}$*-measurable for every* $i \in I$ *and* $\sum_{i \in I} x_i = \sum_{i \in I} e_i\}$ *and* $\boldsymbol{Z}(\mathscr{G}) = \{z \in \prod_{i \in I} L_1(\mu, Y) : z_i$ *is* $\mathscr{G}$*-measurable for every* $i \in I$ *and* $\sum_{i \in I} z_i = 0\}$.

We close this section by discussing the notion of private information measurability of allocations. To say that an agent's allocation is $\mathscr{F}_i$-measurable, it means that his/her consumption is the same in states that he/she cannot distinguish. Also notice that since by assumption initial endowments are $\mathscr{F}_i$-measurable, the net trade of each agents is $\mathscr{F}_i$-measurable as well. This assumption will be dropped in Section 9. However, we believe that this assumption is not only reasonable but it is also tractable from an analytical view point. The example below may be useful to bring out the importance of private measurability.

Example 3.1 Consider an economy with differential information with two agents, one good, and three states (i.e., $\Omega = \{\omega_1, \omega_2, \omega_3\}$) with equal probability (i.e., $\mu(\{\omega\}) = 1/3$ for every $\omega \in \Omega$) where utility functions, initial endowment, and private information sets are given as follows:

$$\begin{aligned} u_1(\omega, x) &= \sqrt{x}, \quad e_1 = (10, 10, 0) \quad \mathscr{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}, \\ u_2(\omega, x) &= \sqrt{x}, \quad e_2 = (10, 0, 10) \quad \mathscr{F}_2 = \{\{\omega_1, \omega_3\}, \{\omega_2\}\}. \end{aligned}$$

In this example, we want to show that without private information measurability, a net trade may not be viable. For simplicity, we only consider the ex ante expected utility. Suppose that agent 1 proposes the net trade $z = (z_1, z_2)$ with

$$z_1 = (-2, -2, 2), \; z_2 = (2, 2, -2).$$

Note that these net trades are not private information measurable. In particular, z_2 is not $\mathscr{F}_2$-measurable. Notice that if state ω_1 is realized, agent 1 may claim that state ω_3 occurred since he/she obtains two units of the good from agent 2 at state ω_3. Observe that agent 2 cannot detect that agent 1 has misreported the state since he/she is not able to distinguish state ω_3 from state ω_1. Conversely, if state ω_1 is realized, agent 2 may claim that state ω_2

occurred since he/she obtains two units of the good from agent 1 at state ω_2 (agent 1 cannot distinguish state ω_2 from state ω_1). Consequently, the non-$\mathscr{F}_i$-measurability of the net trades has created incentive problems and the contract may not take place. In other words, trade may not be viable without private information measurability. As was mentioned in footnote 2, in this example, private measurability is necessary and sufficient for coalitional incentive compatibility. The latter concept is discussed below.

4 Coalitional Bayesian incentive compatibility

When agents have differential information, arbitrary allocations are not generally viable. In particular, arbitrary allocations might not be incentive compatible in the sense that groups of agents may misreport their information without other agents noticing it, and hence achieve different payoffs.

In Krasa-Yannelis (1994), a concept of coalitional incentive compatibility was introduced. For purposes of comparison, we modify their definition in terms of interim expected utility. An allocation $x = e + z \in \boldsymbol{A}$ is *coalitionally Bayesian incentive compatible* if it is not true that there exist coalition S and states $\omega^*, \omega'(\omega^* \neq \omega')$ with $\omega' \in \bigcap_{i \notin S} E_i(\omega^*)$ such that

$$\frac{1}{\mu(E_i(\omega^*))}\int_{E_i(\omega^*)} u_i(\omega, e_i(\omega) + z_i(\omega'))d\mu(\omega)$$
$$> \frac{1}{\mu(E_i(\omega^*))}\int_{E_i(\omega^*)} u_i(\omega, e_i(\omega) + z_i(\omega))d\mu(\omega)$$

for every $i \in S$. Notice that in Krasa-Yannelis (1994), instead of the interim expected utility V_i, the ex post utility function u_i is used. In essence, this concept assures that no coalition S can make redistributions among themselves in states that the complementary coalition cannot distinguish, and become better off. In other words, if state ω^* occurs and the agents in the coalition $I \backslash S$ cannot distinguish between the state ω^* and ω', it must be the case that the agents of coalition S cannot become better off by announcing ω' instead of the actually occurred ω^*. The measurability implies that $\omega' \notin E_i(\omega^*)$ for every agent i in the coalition S.

As in Palfrey-Srivastava (1989), a *deception* for agent i is a function $\alpha_i : \mathscr{F}_i \to \mathscr{F}_i$. Let $\alpha_i^* : \mathscr{F}_i \to \mathscr{F}_i$ be the truth-telling for agent *i*. A deception vector $\alpha = (\alpha_i)_{i \in I}$ is *compatible* with F if $\alpha(\omega) := \cap_{i \in I} \alpha_i(E_i(\omega)) \neq \emptyset$ for every $\omega \in \Omega$. We use the following notation:[9] $\alpha_S^*(\omega) = E^s(\omega) = \cap_{i \in S} E_i(\omega)$, $\alpha_{-s}(\omega) = E^{-S}(\omega) = \cap_{i \notin S} E_i(\omega)$, $\alpha_S(\omega) = E_\alpha^S(\omega) = \cap_{i \in S} \alpha_i(E_i(\omega))$, $\alpha_{-S}(\omega) = E_\alpha^{-S}(\omega) = \cap_{i \notin S} \alpha_i(E_i(\omega))$. Let $z \in Z$ be a feasible net trade. If α is compatible with F, then $(z \circ \alpha)(\omega) = z(\alpha(\omega)) = z(\omega')$ for all $\omega' \in \alpha(\omega)$. Otherwise

[9] For example, consider the following information structure:
$\mathscr{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}, \mathscr{F}_2 = \{\{\omega_1, \omega_3\}, \{\omega_2\}\}, \mathscr{F}_3 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$ Let us define a deception as follows: for every $\omega, \alpha_i(E_i(\omega)) = \{\omega_1\}, \forall_i = 1, 2$ and $\alpha_3(E_3(\omega)) = E_3(\omega)$. Then for the coalition $S = \{1, 3\}, \alpha_S^*(\omega_3) = E^S(\omega_3) = \{\omega_3\}, \alpha_{-S}^*(\omega_3) = E^{-S}(\omega_3) = \{\omega_1, \omega_3\}, \alpha_S(\omega_3) = E_\alpha^S(\omega_3) = \{\omega_1\}$.

$z \circ \alpha = 0$. Note that $(z \circ [\alpha])_i = z_i \circ \alpha$ and $(z \circ [\alpha^*])(\omega) = z(\omega)$. Recall from Lemma 1 of Palfrey-Srivastava (1989, p.120) that for every $i \in I$, if $\omega' \in E_i(\omega)$, then $[\alpha](\omega') \subset E_i(\alpha(\omega))$ for every $i \in I$, where $E_i(\alpha(\omega))$ is the event that contains $[\alpha](\omega)$. In view of this Lemma, we immediately conclude that if $z \in Z$, then $z \circ \alpha \in Z$ for each α compatible with F.

In terms of the deception α_i, one can define Bayesian incentive compatibility. It captures the idea that no agent can improve his utility by using a deception, which is not detected by any other agent. Furthermore, the allocation that is generated by the deception is to be feasible. One can notice the difference between our Bayesian incentive compatibility and the standard Bayesian incentive compatibility [see for example Palfrey-Srivastava (1987)]. This property is well known and considered as a basic requirement for a desirable mechanism in differential information economies. However, it turns out that Bayesian incentive compatibility is not strong enough to play a role as a condition in our model.[10]

Definition 4.1: An allocation $x = e + z \in \boldsymbol{A}$ is said to be *Bayesian incentive compatible* (*BIC*) if for every $i \in I$, for every $\omega \in \Omega$, and for every $\alpha_i : \mathscr{F}_i \to \mathscr{F}_i$ with $(\alpha_i, \alpha^*_{-i})$ compatible with F,

$$V_i(\omega, x_i) \geq V_i(\omega, e_i + (z \circ [\alpha_i, \alpha^*_{-i}])_i),$$

where $e + z \circ [\alpha_i, \alpha^*_{-i}] \in \boldsymbol{A}$.

Definition 4.2: An allocation $x = e + z \in \boldsymbol{A}$ is said to be *coalitionally Bayesian incentive compatible* (*CBIC*) if it is not true that there exists a state $\omega \in \Omega$, a coalition $S \subset I$, and a deception $\alpha_S : \prod_{i \in S} \mathscr{F}_i \to \prod_{i \in S} \mathscr{F}_i$ such that $(\alpha_S, \alpha^*_{-S})$ is compatible with F and for every $i \in S$,

$$V_i(\omega, e_i + (z \circ [\alpha_S, \alpha^*_{-S}])_i) > V_i(\omega, x_i),$$

where $e + z \circ [\alpha_S, \alpha^*_{-S}] \in \boldsymbol{A}$.

This notion of incentive compatibility states that it is not possible for any coalition S to become better off by announcing a false event[11]. Observe that if S is a singleton, then the CBIC condition is reduced to standard BIC condition. This implies that coalitional Bayesian incentive compatibility is a stronger condition than Bayesian incentive compatibility.

Definition 4.3: An allocation $x = e + z \in \boldsymbol{A}$ is said to be *weakly coalitional Bayesian incentive compatible* (*weakly CBIC*) if it is not true that there exists a state $\omega \in \Omega$, a coalition $S \subset I$, and a deception $\alpha_S : \prod_{i \in S} \mathscr{F}_i \to \prod_{i \in S} \mathscr{F}_i$ such that for every $i \in S, \alpha_i(E_i(\omega')) = E_i(\omega') \forall \omega' \notin E_i(\omega), E_i(\omega) \in \wedge_{i \in S} \mathscr{F}_i$, and

$$V_i(\omega, e_i + (z \circ [\alpha_S, \alpha^*_{-S}])_i) > V_i(\omega, x_i),$$

[10] See Theorem 6.4.

[11] Note that whenever $E_i(\omega)$ and $\alpha_i(E_i(\omega))$ are singletons for every $i \in S$, our notion coincides with that of Krasa-Yannelis (1994), provided that ex post utility functions are used.

where $e + z \circ [\alpha_S, \alpha^*_{-S}] \in \boldsymbol{A}$.

This is obtained from adding $E_i(\omega) \in \bigwedge_{i \in S} \mathscr{F}_i, \forall_i \in S$ to the deceiving conditions in CBIC. In particular, the true event, which is misreported, is common knowledge to the deceiving coalition.

We can now define a much stronger notion of incentive compatibility.

Definition 4.4: An allocation $x = e + z \in \boldsymbol{A}$ is said to be *strongly coalitional Bayesian incentive compatible* (*SCBIC*) if it is not true that there exist a state $\omega \in \Omega$, a coalition $S \subset I$, and a deception $\alpha_S : \prod_{i \in S} \mathscr{F}_i \to \prod_{i \in S} \mathscr{F}_i$ such that $(\alpha_S, \alpha^*_{-S})$ is compatible with F and for every $i \in S$,

$$V_i(\omega, e_i + (z \circ [\alpha_S, \alpha^*_{-S}])_i) \geq V_i(\omega, x_i)$$

with strict inequality for some $i \in S$, where $e + z \circ [\alpha_S, \alpha^*_{-S}] \in \boldsymbol{A}$.

Definition 4.5: An allocation $x = e + z \in \boldsymbol{A}$ is said to be *t-coalitional Bayesian incentive compatible* (*TCBIC*)[12] if it is not true that there exist a state $\omega \in \Omega$, a coalition $S \subset I$, and a deception $\alpha_S : \prod_{i \in S} \mathscr{F}_i \to \prod_{i \in S} \mathscr{F}_i$, and a transfer $(t_i)_{i \in S} \in \prod_{i \in S} L_i$ with $\sum\limits_{i \in S} t_i = 0$, *each t_i is* $\bigwedge\limits_{i \in S} \mathscr{F}_i$-measurable such that $(\alpha_S, \alpha^*_{-S})$ is compatible with F and for every $i \in S$,

$$V_i(\omega, e_i + (z \circ [\alpha_S, \alpha^*_{-S}])_i + t_i) > V_i(\omega, x_i),$$

where $e + z \circ [\alpha_S, \alpha^*_{-S}] \in \boldsymbol{A}$.

The t-coalitional Bayesian incentive compatibility models the idea that it is impossible for any coalition to cheat the complementary coalition by misreporting the event and making side payments to each other which cannot be observed by agents who are not members of this coalition. When $u_i(\omega, \cdot)$ is monotone and continuous for every $i \in I$ and $\omega \in \Omega$, one can easily show that the notion of SCBIC is equivalent to the TCBIC. In particular, if an allocation is not SCBIC, the agent who became strictly better off can make side payments to every agent in the deceiving coalition and make them strictly better off.

By observing the definitions, one can easily check that the following relationship between these concepts of incentive compatibility holds:

$$TCBIC \Rightarrow SCBIC \Rightarrow CBIC \Rightarrow \textit{weakly } CBIC \Rightarrow BIC.$$

5 Efficiency

5.1 Efficiency concepts in differential information economies

The notions of informational efficiency discussed below are distinguished depending the degree of private information. The ex ante efficiency is defined

[12] This is an interim version of the *strong coalitional incentive compatibility* of Krasa-Yannelis (1994).

at the stage where every agent has private information but no state is yet realized. The interim efficiency is defined at the stage where every agent knows his/her private information event which contains the realized state. The ex post efficiency is defined at the stage where every agent has complete information. Because the interim stage and the ex post stage depend on states, it is more difficult to define the notions of efficiency. In particular, for the definition of interim efficiency, the possibility of communication between all the agents when they block the proposed allocation make the problem even harder. In order to address the possibility of communication among agents, we will introduce more notation.

Denote by $\bigwedge_{i\in I}\mathscr{F}_i$ the *finest common coarsening* of $\{\mathscr{F}_i : i\in I\}$, *i.e.*, the finest partition of Ω which is coarser than $\mathscr{F}_i$ for every $i\in I$. An event E is said to be *common knowledge at* ω if $(\bigwedge_{i\in I}\mathscr{F}_i)(\omega)\subset E$ where $(\bigwedge_{i\in I}\mathscr{F}_i)(\omega)$ is the event of $\bigwedge_{i\in I}\mathscr{F}_i$ containing ω. Notice that $(\bigwedge_{i\in I}\mathscr{F}_i)(\omega)$ itself is common knowledge at ω. We also call $\bigwedge_{i\in I}\mathscr{F}_i$ the *common knowledge partitions* of Ω. Denote by $\bigvee_{i\in I}\mathscr{F}_i$ coarsest common refinement of $\{\mathscr{F}_i : i\in I\}$, *i.e.*, the coarsest partition of Ω which is finer than $\mathscr{F}_i$ for every $i\in I$. Denote by $(\bigvee_{i\in I}\mathscr{F}_i)(\omega)$ the event of $\bigvee_{i\in I}\mathscr{F}_i$ containing ω. We also call $\bigvee_{i\in I}\mathscr{F}_i$ the *pooled information partition.*[13]

Several notions of efficiency will be defined below. The main differences of these concepts are basically two. Firstly, the degree of information sharing of the grand coalition, *i.e.*, do agents make redistribution of their initial endowment based on their own private information, common knowledge information, or pooled information? Secondly, what kind of expected utility is used, i.e., interim, ex ante, or ex post?

5.2 Ex ante efficiency

The notion of ex ante efficiency is defined in terms of the ex ante expected utility. If the grand coalition of agents is allowed to redistribute their resources among themselves to become better off by using the common knowledge information, the ex ante coarse efficiency is a natural concept of efficiency.

Definition 5.2.1: An allocation $x\in \boldsymbol{A}$ is *ex ante coarse efficient* if there is no $x'\in \boldsymbol{A}$ such that $x'-e\in Z(\bigwedge_{i\in I}\mathscr{F}_i)$ and $\overline{V}_i(x_i')>\overline{V}_i(x_i)$ for every $i\in I$.

If it is possible for the grand coalition of agents to redistribute their initial endowments among themselves to become better off by using their own private information, the ex ante private efficiency can be adopted.

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[13] In the context of σ-algebra, $\wedge_{i\in I}\mathscr{F}_i$ denotes the *meet, i.e.* the maximal (finest) σ-algebra contained in every σ-algebra $\mathscr{F}_i$ and $\vee_{i\in I}\mathscr{F}_i$ denotes the *join, i.e.*, the minimal (coarsest) σ-algebra containing every σ-algebra $\mathscr{F}_i$.

Definition 5.2.2: All allocation $x \in \boldsymbol{A}$ is *ex ante private efficient*[14] there is no $x' \in \boldsymbol{A}$ such that $\overline{V}_i(x_i') > \overline{V}_i(x_i)$ for every $i \in I$.

If it is possible for the grand coalition of agents to redistribute their initial endowments among themselves to become better off by pooling and sharing their private information, the ex ante fine efficiency can be defined as follows.

Definition 5.2.3: An allocation $x \in \boldsymbol{A}$ is *ex ante fine efficient* if there is no $x' \in \boldsymbol{A}(\bigvee_{i\in I}\mathscr{F}_i)$ such that $\overline{V}_i(x_i') > \overline{V}_i(x_i)$ for every $i \in I$.

In addition, if a feasible allocation is allowed to be measurable with respect to the pooled information, then a weaker concept can be defined.

Definition 5.2.4: An allocation $x \in \boldsymbol{A}(\bigvee_{i\in I}\mathscr{F}_i)$ is *ex ante weak fine efficient* if there is no $x' \in \boldsymbol{A}(\bigvee_{i\in I}\mathscr{F}_i)$ such that $\overline{V}_i(x_i') > \overline{V}_i(x_i)$for every $i \in I$.

5.3 Interim efficiency

The interim efficiency notions below will be defined in terms of the interim expected utility. If the grand coalitian of agents can redistribute their resources among themselves to become better off by using the common knowledge information, the interim coarse efficiency can be defined as follows.

Definition 5.3.1: An allocation $x \in \boldsymbol{A}$ is *interim coarse efficient* if there is no $x' \in \boldsymbol{A}$ such that $x' - e \in Z(\bigwedge_{i\in I}\mathscr{F}_i)$ and for some $\omega \in \Omega$, $V_i(\omega, x_i') > V_i(\omega, x_i)$ for every $i \in I$.

If it is possible for the grand coalition of agents to redistribute their initial endowments among themselves to become better off by using their own private information, the interim private efficiency can be defined as follows.

Definition 5.3.2: An allocation $x \in \boldsymbol{A}$ is *interim private efficient*[15] if there is no $x' \in \boldsymbol{A}$ such that for some $\omega \in \Omega$, $V_i(\omega, x_i') > V_i(\omega, x_i)$ for every $i \in I$.

If it is possible for the grand coalition of agents to redistribute their initial endowments among themselves to become better off by pooling and sharing their information, the interim fine efficiency can be defined as follows.

Definition 5.3.3: An allocation $x \in \boldsymbol{A}$ is *interim fine efficient* if there is no $x' \in \boldsymbol{A}(\bigvee_{i\in I}\mathscr{F}_i)$ such that for some $\omega \in \Omega$, $V_i(\omega, x_i') > V_i(\omega, x_i)$ for every $i \in I$.[16]

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[14] Notice that if $u_i(\omega, \cdot)$ is continuous and monotone, this definition is equivalent to: An allocation $x \in \boldsymbol{A}$ is *strongly ex ante private efficient* if there is no $x' \in \boldsymbol{A}$ such that $\overline{V}_i(x_i') \geq \overline{V}_i(x_i)$ for every $i \in I$ with strict *inequality for some* $\mathbf{i} \in I$.

[15] An allocation $x \in \boldsymbol{A}$ is *strongly interim private efficient* if there is no $x' \in \boldsymbol{A}$ such that for some $\omega \in \Omega$, $V_i(\omega, x_i') \geq V_i(\omega, x_i)$ for every $i \in I$ with strict inequality for some $i \in I$.

[16] One may consider an interim expected utility which takes into account the pooled information. But throughout the paper, we ignore this effect.

If the feasible allocation is allowed to be measurable w.r.t to the pooled information, then a weaker concept can be defined as follows.

Definition 5.3.4: An allocation $x \in \boldsymbol{A}\,(\bigvee_{i\in I}\mathscr{F}_i)$ is *interim weak fine efficient* if there is no $x' \in \boldsymbol{A}(\bigvee_{i\in I}\mathscr{F}_i)$ such that for some $\omega \in \Omega$, $V_i(\omega, x_i') > V_i(\omega, x_i)$ for every $i \in I$.

Moreover, if the event where every agent becomes better off is common knowledge to the grand coalition, then the notion of weakly interim efficiency can be defined as follows.

Definition 5.3.5: An allocation $x \in \boldsymbol{A}$ is *weakly interim efficient* if there is no $x' \in \boldsymbol{A}$ such that for some $E \in \bigwedge_{i\in I}\mathscr{F}_i$, $V_i(\omega, x_i') > V_i(\omega, x_i)$ for every $\omega \in E$ and for every $i \in I$.

Since interim efficiency does depend on states, we have one more notion of interim efficiency, *HM interim efficiency,* which is widely used as interim efficiency notion in economics literature [for example, see Holmström-Myerson (1983, p.1805)].

Definition 5.3.6: An allocation $x \in \boldsymbol{A}$ is *HM interim efficient*[17] if there is no $x' \in \boldsymbol{A}$ such that $V_i(\omega, x_i') \geq V_i(\omega, x_i)$ for every $\omega \in \Omega$ and for every $i \in I$ with strict inequality for some $\omega \in \Omega$ and for some $i \in I$.

In the same way as in interim efficiency, one can define *strongly ex post efficiency, ex post efficiency,* and *HM ex post efficiency* by using ex post utility u_i and ex post feasible set $\boldsymbol{A}^0$.

5.4 Relationship of the efficiency concepts

In economies with certainty, it is known that if the preferences are monotone and continuous, strong efficiency and efficiency are equivalent. In the same way, one could get corresponding equivalence[18] for differential information economies. Furthermore, one can easily prove that efficiency concepts are stronger if the information sharing of the grand coalition is finer in either ex ante or interim case as the following proposition indicate:

Proposition 5.4.1: The following statements hold.

(a) Every ex ante fine efficient allocation in $\mathscr{E}$ is also ex ante private efficient.
(b) Every ex ante private efficient allocation in $\mathscr{E}$ is also ex ante coarse efficient.
(c) Every ex ante fine efficient allocation in $\mathscr{E}$ is also ex ante weak fine efficient.

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[17] This is different from that of Homström-Myerson (1983) in that they do not impose the private information measurability.

[18] Assume that $u_i(\omega,)$ is monotone and continuous for every $i \in I$ and $\omega \in \Omega$. By simply observing the definitions, one can easily check that an allocation is strongly interim private (strongly ex ante private, strongly ex post, resp.) efficient if and only if it is interim private (ex ante private, ex post, resp.) efficient.

Proof: (a) Let x be an ex ante fine efficient allocation. Suppose that it is not ex ante private efficient. Then there exist $x' \in \boldsymbol{A}$ such that $\overline{V}_i(x_i') > \overline{V}_i(x_i)$ for every $i \in I$. Since $\boldsymbol{A} \subset \boldsymbol{A}(\bigvee_{i\in I} \mathscr{F}_i)$, we have $x' \in \boldsymbol{A}(\bigvee_{i\in I} \mathscr{F}_i)$ such that $\overline{V}_i(x_i') > \overline{V}_i(x_i)$ for every $i \in I$, a contradiction.

(b) Let x be an ex ante private efficient allocation. Suppose that it is not ex ante coarse efficient. Then there exists $x' \in \boldsymbol{A}$ such that $x' - e \in Z(\bigwedge_{i\in I} \mathscr{F}_i)$ and $\overline{V}_i(x_i') > \overline{V}_i(x_i)$ for every $i \in I$. Since $Z(\bigwedge_{i\in I} \mathscr{F}_i) \subset Z$,we have $x' \in \boldsymbol{A}$ such that $\overline{V}(x_i') > \overline{V}_i(x_i)$ for every $i \in I$, a contradiction.

(c) Let $x \in \boldsymbol{A}$ be an ex ante fine efficient allocation. Suppose that it is not ex ante weak fine efficient. Then there exists $x' \in \boldsymbol{A}(\bigvee_{i\in I} \mathscr{F}_i)$ such that $\overline{V}_i(x_i') > \overline{V}_i(x_i)$ for every $i \in I$. Since $\boldsymbol{A} \subset \boldsymbol{A}(\bigvee_{i\in I} \mathscr{F}_i)$, we have $x \in \boldsymbol{A}(\bigvee_{i\in I} \mathscr{F}_i)$ such that there exists $x' \in \boldsymbol{A}(\bigvee_{i\in I} \mathscr{F}_i)$ such that $\overline{V}(x_i') > \overline{V}_i(x_i)$ for every $i \in I$, a contradiction. □

Applying the same arguments about the information sharing, we get the same results for the concepts of interim efficiency.

Proposition 5.4.2: The following statements hold.

(a) Every interim fine efficient allocation in $\mathscr{E}$ is also interim private efficient.
(b) Every interim private efficient allocation in $\mathscr{E}$ is also interim coarse efficient.
(c) Every interim fine efficient allocation in $\mathscr{E}$ is also interim weak fine efficient.

Proof: Follow the argument adopted for the proof of Proposition 5.4.1. □

Proposition 5.4.3: The following statements hold.

(a) Every interim private efficient allocation in $\mathscr{E}$ is also weakly interim efficient.
(b) Every HM interim efficient allocation in $\mathscr{E}$ is also weakly interim efficient.
(c) If $u_i(\omega, \cdot)$ is monotone and continuous, then every interim private efficient allocation in $\mathscr{E}$ is also HM interim efficient.

Proof: (a) Let x be an interim efficient allocation. Suppose that x is not weakly interim efficient. Then there is a feasible allocation x' such that for some common knowledge event $E \in \bigwedge_{i\in I} \mathscr{F}_i$, $V_i(\omega, x_i') > V_i(\omega, x_i)$ for every $\omega \in E$ and for every $i \in I$. This implies that for some $\omega \in \Omega$, $V_i(\omega, x_i') > V_i(\omega, x_i)$ for every $i \in I$. Hence, x is not interim efficient, a contradiction.

(b) Let x be a HM interim efficient allocation. Suppose that x is not weakly interim efficient. Then there is a feasible allocation x' such that for some common knowledge event $E \in \bigwedge_{i\in I} \mathscr{F}_i$, $V_i(\omega, x_i') > V_i(\omega, x_i)$ for every $\omega \in E$ and for every $i \in I$. Consider a new allocation $x^* = (x_i^*)_{i\in I} \in \boldsymbol{A}$, where

$$x_i^*(\omega') = \begin{cases} x_i'(\omega') & \text{if } \omega' \in E, \\ x_i(\omega') & \text{otherwise.} \end{cases}$$

It follows that $V_i(\omega, x_i^*) \geq V_i(\omega, x_i)$ for every $\omega \in \Omega$ and for every $i \in I$. Moreover, $V_i(\omega, x_i^*) > V_i(\omega, x_i)$ for some $\omega \in \Omega$ and for some $i \in I$. Hence, x is not HM interim efficient, a contradiction.

(c) Let x be an interim private efficient allocation. Suppose that x is not HM interim efficient. Then there is a feasible allocation x' such that, $V_i(\omega, x'_i) \geq V_i(\omega, x_i)$ for every $\omega \in \Omega$ and for every $i \in I$ with strict inequality for some $\omega^* \in \Omega$ and for some $k \in I$. Since for each $i \in I$ and for each fixed $\omega \in \Omega$, $V_i(\omega, \cdot)$ is continuous by Lemma 3.1.1, there is an $\varepsilon > 0$ such that that $V_k(\omega^*, x'_k - \varepsilon \cdot 1 > V_k(\omega^*, x_k)$. Consider a new allocation $x^* = (x^*_i)_{i \in I} \in \boldsymbol{A}$ with

$$x^*_i = \begin{cases} x'_i - \varepsilon \cdot \mathbf{1} & \text{if } i = k, \\ x'_i + \frac{1}{|I|-1} \varepsilon \cdot \mathbf{1} & \text{otherwise.} \end{cases}$$

Since $V_i(\omega, \cdot)$ is monotone, it follows that $V_i(\omega^*, x^*_i) > V_i(\omega^*, x_i)$ for every $i \in I$. Hence, x is not interim private efficient, a contradiction. □

Recall that the ex ante expected utility and the interim expected utility are related in the following way:

$$\overline{V}_i(x_i) = \sum_{E_i(\omega) \in \mathscr{F}_i} \mu(E_i(\omega)) V_i(\omega, x_i). \tag{5.4.1}$$

This gives the relationship between ex ante private efficiency and HM interim efficiency.

Proposition 5.4.4: Assume that $u_i(\omega, \cdot)$ is monotone and continuous for every $i \in I$ and $\omega \in \Omega$. Every ex ante efficient allocation in $\mathscr{E}$ is also HM interim efficient.[19]

Proof: Let x be an ex ante private efficient allocation. Suppose that x is not HM interim efficient. Then there is an feasible allocation x' such that $V_i(\omega, x'_i) \geq V_i(\omega, x_i)$ for every $\omega \in \Omega$ and for every $i \in I$ with strict inequality for some $\omega^* \in \Omega$ and some $k \in I$. It follows from (5.4.1) that $\overline{V}_i(x'_i) \geq \overline{V}_i(x_i)$ for every $i \in I$ with strict inequality for $k \in I$. Since $\overline{V}_i$ is continous (recall Lemma 3.1.1), there is an $\varepsilon > 0$ such that $\overline{V}_k(x'_k - \varepsilon \cdot \mathbf{1}) > \overline{V}_k(x_k)$. Consider a new allocation $x^* = (x^*_i)_{i \in I} \in \boldsymbol{A}$ where

$$x^*_i = \begin{cases} x'_i - \varepsilon \cdot \mathbf{1} & \text{if } i = k, \\ x'_i + \frac{1}{|I|-1} \varepsilon \cdot \mathbf{1} & \text{otherwise.} \end{cases}$$

Since $\overline{V}_i$ is monotone, $\overline{V}_i(x^*_i) > \overline{V}_i(x_i)$ for every $i \in I$. This implies that x is not ex ante private efficient, a contradiction. □

Corollary 5.4.5: Assume that $u_i(\omega, \cdot)$ is monotone and continuous for every $i \in I$ and $\omega \in \Omega$. Every ex ante private efficient allocation in $\mathscr{E}$ is weakly interim efficient.

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[19] When the private information measurability is not imposed, one can show that the strong ex ante efficiency implies the HM interim efficiency, which in turn implies the HM ex post efficiency. This is a well-known fact [see Holmström-Myerson (1983)]. For comparison, let T_i be the type space of agent i. Then $E_i(t) = \{t_i\} \times T_{-i}$ with $t = (t_i, t_{-i})$. Thus, in the context of type representation of private information, the private information measurability is described by $x_i(t_i, t_{-i}) = x_i(t_i, t'_{-i})$ for every t_{-i} and t'_{-i} in T_{-i} since $E_i(t_i, t_{-i}) = E_i(t_i, t'_{-i})$.

Proof: It follows from Proposition 5.4.3 (b) and Proposition 5.4.4. □

Unlike Holmström-Myerson (1983), it turns out that there is no direct implication between the ex ante private efficiency and the interim private efficiency, as the proposition below indicates.

Proposition 5.4.6: An ex ante private efficient (weakly interim efficient, or HM interim efficient) allocation in $\mathscr{E}$ may not be interim private efficient.

Proof: Consider an economy with differential information with three agents, two goods, and three equally probable states, where utility functions, random initial endowments, and private information sets are given as follows:

$$u_1(\omega, x^1, x^2) = \sqrt{x^1 x^2}, e_1 = ((10,0),(10,0),(10,0)), \mathscr{F}_1 = \{\{\omega_1, \omega_2, \omega_3\}\},$$
$$u_2(\omega, x^1, x^2) = \sqrt{x^1 x^2}, e_2 = ((4,4),(1,5),(1,5)), \quad \mathscr{F}_2 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\},$$
$$u_3(\omega, x^1, x^2) = \sqrt{x^1 x^2}, e_3 = ((0,1),(1,3),(3,4)), \quad \mathscr{F}_3 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}.$$

The allocation $x = (x_1, x_2, x_3)$ with

$$x_1 = ((6,2),(6,2),(6,2)),$$
$$x_2 = ((7,3),(5,3),(5,3)),$$
$$x_3 = ((1,0),(1,3),(3,4))$$

is an ex ante private (weakly interim efficient, or HM interim efficient) but is not interim private efficient, since the allocation $x' = (x'_1, x'_2, x'_3)$ with

$$x'_1 = ((6.1,2),(6.1,2),(6.1,2)),$$
$$x'_2 = ((7,3),(4,4),(4,4)),$$
$$x'_3 = ((0.9,0),(1.9,2),(0.9,3))$$

results in $V_i(\omega_2, x'_i) > V_i(\omega_2, x_i)$ for every $i \in I$. □

Denote by $\overline{\mathscr{E}}$ an economy as defined in Section 3, with the only difference that now $Y_+ = \boldsymbol{R}_+$, *i.e.*, we have one good per state. In this case, the set of feasible allocations lies in the infinite dimensional space $L_1(\mu, \boldsymbol{R}_+)$. In a one good economy, the set of feasible allocations is equivalent to the set of interim efficient allocations. It is obvious that every interim efficient allocation is feasible. The other direction is clear too. Indeed, from a given feasible allocation, a change to any other feasible allocation makes at least one agent become worse off at some state because there is only one good. It can be proved formally as follows.

Proposition 5.4.7: Every feasible allocation in $\overline{\mathscr{E}}$ is interim coarse efficient.

Proof: Suppose that a feasible allocation $x \in \boldsymbol{A}$ is not interim coarse efficient. Then there exist a state $\omega \in \Omega$, a agent $i \in I$, an allocation $x' \in \boldsymbol{A}$ such that $x' - e \in Z(\bigwedge_{i \in I} \mathscr{F}_i)$ and for every $i \in I$,

$$V_i(\omega, x_i') > V_i(\omega, x_i). \tag{5.4.2}$$

Since there is only one good, (5.4.2) implies that for every $i \in I$, $x_i'(\omega) > x_i(\omega)$ by monotonicity and measurability. Hence, $\overline{e}(\omega) = \sum_{i\in I} x_i'(\omega) > \sum_{i\in I} x_i(\omega) = \overline{e}(\omega)$, a contradiction. □

Corollary 5.4.8: Every feasible allocation in $\bar{\mathscr{E}}$ is interim private efficient.

Proof: Since every interim coarse efficient allocation is interim private efficient [Proposition 5.4.2(b)], the conclusion follows from Proposition 5.4.7. □

6 Relationship of efficiency with incentive compatibility

It is well-known that the Bayesian incentive compatibility condition is too restrictive for achieving socially desirable allocations. In particular, Myerson (1979) recognized that most interim efficient allocations may not be Bayesian incentive compatible. However, if a simple condition is assumed (that is, the private information measurability), the BIC condition turns out to be so weak that every feasible allocation is BIC. However, the CBIC condition seems more appropriate and one can show that several efficiency concepts defined in Section 5 are always coalitionally Bayesian incentive compatible.

Proposition 6.1: Every interim coarse efficient allocation in $\mathscr{E}$ is TCBIC.

Proof: Suppose that $x = e + z \in \boldsymbol{A}$ is interim coarse efficient but it is not TCBIC. Then there exists a state $\omega^* \in \Omega$, a coalition S, a deception $\alpha_S : \prod_{i\in S} \mathscr{F}_i \to \prod_{i\in S} \mathscr{F}_i$, and a transfer $(t_i)_{i\in S} \in \prod_{i\in S} L_i$ with $\sum_{i\in S} t_i = 0$, each t_i is $\bigwedge_{i\in S} \mathscr{F}_i$-measurable and such that for every $i \in S$,

$$V_i(\omega^*, e_i + (z \circ [\alpha_S, \alpha^*_{-S}])_i + t_i) > V_i(\omega^*, x_i),$$

where $e + z \circ [\alpha_S, \alpha^*_{-S}] \in \boldsymbol{A}$. Since for every $\omega' \in E^S_\alpha(\omega^*) \cap E^{-S}(\omega^*)$ it holds that $z_i(\omega') = z_i(\omega^*)$, *i.e.*, $(z \circ [\alpha_S, \alpha^*_{-S}])_i(\omega^*) = z_i(\omega^*)$ for every $i \notin S$, it must be the case that for every $i \notin S$,

$$V_i(\omega^*, e_i + (z \circ [\alpha_S, \alpha^*_{-S}])_i) = V_i(\omega^*, x_i). \tag{6.1}$$

Since for each $i \in I$ and for each fixed $\omega \in \Omega$, $V_i(\omega, \cdot)$ is continuous by Lemma 3.1.1, there exists an $\varepsilon > 0$ such that for every $i \in S$,

$$V_i(\omega^*, e_i + z \circ [\alpha_S, \alpha^*_{-S}]_i + t_i - \varepsilon \cdot \mathbf{1}) > V_i(\omega^*, x_i). \tag{6.2}$$

Let us define $z' = (z_i')_{i\in I} : \Omega \to \hat{Z}$ by $z_i'(\omega) = (z_i \circ [\alpha_S, \alpha^*_{-S}])(\omega^*) + t_i(\omega^*)$ for every $\omega \in \Omega$, where $t_i = 0$ for every $i \notin S$. Define $x' = (x_i')_{i\in I}$ by

$$x_i' = \begin{cases} e_i + z_i' - \varepsilon \cdot \mathbf{1} & \text{if } i \in S, \\ e_i + z_i' + \frac{|S|}{|I \backslash S|} \varepsilon \cdot \mathbf{1} & \text{if } i \notin S. \end{cases}$$

Note that $x_i' - e_i$ is $\bigwedge_{i\in I} \mathscr{F}_i$-measurable and x' is a feasible allocation since $\sum_{i\in I} (z \circ [\alpha_S, \alpha^*_{-S}])_i = 0$. However, (6.2) implies that $V_i(\omega^*, x_i') > V_i(\omega^*, x_i)$ for every $i \in S$. Because $V_i(\omega, \cdot)$ is monotone for every $i \in I$, (6.1) implies that

$V_i(\omega^*, x_i') > V_i(\omega^*, x_i)$ for every $i \notin S$. Hence we have a contradiction to the fact that x is interim coarse efficient. □

Corollary 6.2: Every interim coarse efficient allocation in $\mathscr{E}$ is CBIC.

Proof: Since TCBIC implies CBIC, the conclusion follows from Proposition 6.1. □

Corollary 6.3: Every interim private efficient allocation in $\mathscr{E}$ is CBIC.

Proof: Since every interim private efficient allocation is interim coarse efficient [Proposition 5.4.2 (b)], Corollary 6.2 leads to the assertion. □

Since CBIC implies BIC, we can therefore obtain the following from Corollary 6.3.

Corollary 6.4: Every interim private efficient allocation in $\mathscr{E}$ is BIC.

Proposition 6.5: Assume that $u_i(\omega, \cdot)$ is monotone and continuous for every $\omega \in \Omega$ and for every $i \in I$. Then every weakly interim efficient allocation in ε is weakly CBIC.

Proof: Suppose $x = e + z \in \boldsymbol{A}$ is weakly interim efficient but it is not weakly CBIC. Then there exist a state $\omega^* \in \Omega$, a coalition S, and a deception $\alpha_S : \prod_{i\in S} \mathscr{F}_i \to \prod_{i\in S} \mathscr{F}_i$ such that for every $i \in S, \alpha_i(E_i(\omega')) = E_i(\omega')$ $\forall \omega' \notin E_i(\omega^*)$, $E_i(\omega^*) \in \bigwedge_{i\in S} \mathscr{F}_i$, and

$$V_i(\omega^*, e_i + (z \circ [\alpha_S, \alpha^*_{-S}])_i) > V_i(\omega^*, x_i),$$

where $e + z \circ [\alpha_S, \alpha^*_{-S}] \in \boldsymbol{A}$. Since for every $\omega' \in E^S_\alpha(\omega^*) \cap E^{-S}(\omega^*)$ it holds that $z_i(\omega') = z_i(\omega^*), i.e., (z \circ ([\alpha_S, \alpha^*_{-S}])_i(\omega^*) = z_i(\omega^*)$ for every $i \notin S$, it must be the case that for every $i \notin S$,

$$V_i(\omega', e_i + (z \circ [\alpha_S, \alpha^*_{-S}])_i) = V_i(\omega^*, x_i). \tag{6.3}$$

Since for each $\omega \in \Omega$, $V_i(\omega, \cdot)$ is continuous for every $i \in I$ by Lemma 3.1.1, there exists an $\varepsilon > 0$ such that for every $\omega \in \Omega$ and for every $i \in S$

$$V_i(\omega, e_i + (z \circ [\alpha_S, \alpha^*_{-S}])_i - \varepsilon \cdot \mathbf{1}) > \mathbf{V_i}(\omega, \mathbf{x_i}). \tag{6.4}$$

Let us define $x' = (x_i')_{i\in I}$ by

$$x_i'(\omega') = \begin{cases} e_i + (z \circ [\alpha_S, \alpha^*_{-S}])_i - \varepsilon \cdot \mathbf{1} & \text{if } i \in S, \\ x_i + \frac{|S|}{|I \setminus S|} \varepsilon \cdot \mathbf{1} & \text{otherwise.} \end{cases}$$

Note that $x' \in \boldsymbol{A}$. (6.4) implies that $V(\omega, x_i') > V_i(\omega, x_i)$ for every $\omega \in E$ and for every $i \in S$. Because $V_i(\omega, .)$ is monotone for every $\omega \in \Omega$ and for every $i \in I$, (6.3) means that $V_i(\omega, x_i') > V_i(\omega, x_i)$ for every $\omega \in \Omega$ and for every $i \notin S$. Hence x is not weakly interim efficient, a contradiction. □

Corollary 6.6: Assume that $u_i(\omega, \cdot)$ is monotone and continuous for every $\omega \in \Omega$ and for every $i \in I$. Then every HM interim efficient allocation in $\mathscr{E}$ is weakly CBIC.

Proof: It follows from Proposition 5.4.3 (b) and Proposition 6.5. $\square$

Corollary 6.7: Assume that $u_i(\omega, \cdot)$ is monotone and continuous for every $\omega \in \Omega$ and for every $i \in I$. Then every ex ante private efficient allocation in $\mathscr{E}$ is weakly CBIC.

Proof: It follows from Corollary 5.4.5 and Proposition 6.5. $\square$

Proposition 6.8: A weakly interim efficient allocation in $\mathscr{E}$ may not be CBIC.

Proof: Consider the same economy as in Proposition 5.4.5. The allocation $x = (x_1, x_2, x_3)$ with

$$\begin{aligned} x_1 &= ((6,2),(6,2),(6,2)),\\ x_2 &= ((7,3),(5,3),(5,3)),\\ x_1 &= ((1,0),(1,3),(3,4)) \end{aligned}$$

is weakly interim efficient allocation but is not interim private efficient allocation. However, the allocation x is not coalitional Bayesian incentive compatible, since, at ω_2, coalition $S = \{2,3\}$ with a deception $\alpha_i(E_i(\omega)) = \{\omega_1\}$ for every $\omega \in \Omega$ and $i \in S$ will make its members better off, *i.e.*,

$$\begin{aligned} V_2(\omega_2, e_2 + (z \circ [\alpha_S, \alpha_1^*])_2) &> V_2(\omega_2, x_2),\\ V_3(\omega_2, e_3 + (z \circ [\alpha_S, \alpha_1^*])_3) &> V_3(\omega_2, x_3). \end{aligned}$$ $\square$

Corollary 6.9: A HM interim efficient allocation in $\mathscr{E}$ may not be CBIC.

In fact, we can show that any feasible allocation is Bayesian incentive compatible. This means that the Bayesian incentive compatibility is too weak to play a role as a condition.

Proposition 6.10: Every feasible allocation in $\mathscr{E}$ is BIC.

Proof: Suppose a feasible allocation $x \in \boldsymbol{A}$ is not BIC. Then there exist a state $\omega \in \Omega$, an agent $i \in I$, and a deception $\alpha_i : \mathscr{F}_i \to \mathscr{F}_i$ such that

$$V_i(\omega, e_i + (z \circ [\alpha_i, \alpha^*_{-i}])_i) > V_i(\omega, x_i), \tag{6.5}$$

where $e + z \circ [\alpha_i, \alpha^*_{-i}] \in \boldsymbol{A}$. Since for every $\omega' \in E^i_\alpha(\omega) \cap E^{-i}(\omega)$ it holds that $z_k(\omega') = z_k(\omega)$, *i.e.*, $(z \circ [\alpha_i, \alpha^*_{-i}])_k(\omega) = z_k(\omega)$ for every $k \neq i$, it follows from the feasibility that

$$(z \circ [\alpha_i, \alpha^*_{-i}])_i(\omega) = z_i(\omega).$$

By measurability, we obtain

$$V_i(\omega, e_i + (z \circ [\alpha_i, \alpha^*_{-i}])_i) = V_i(\omega, x_i),$$

a contradiction to (6.5). $\square$

It is worth noting that in an economy $\overline{\mathscr{E}}$ with one good per state, the set of interim private efficient allocations coincides with the set of Bayesian incentive compatible allocations. This can be shown by combining Corollary 5.4.8 with Proposition 6.10.

Corollary 6.11: The set of interim private efficient allocations in $\overline{\mathscr{E}}$, the set of Bayesian incentive compatible allocations $\overline{\mathscr{E}}$, and the set of feasible allocations $\overline{\mathscr{E}}$, are all equivalent.

Since every BIC allocation is feasible, Proposition 6.10 implies that the set of feasible allocations is equivalent to the set of BIC allocations. Apparently, our result looks contradicting that of Myerson (1979), *i.e.*, an interim efficient allocation may not be Bayesian incentive compatible. Note that the interim efficiency of Myerson (1979) is equivalent to our HM interim efficiency except the private information measurability assumption on allocations. In view of Proposition 5.4.3 (c), our interim efficiency is stronger than that of Myerson (1979). As it will be shown with an example below, Myerson's argument is robust without the imposition of private information measurability (*i.e.*, an interim efficient allocation may not be Bayesian incentive compatible). This is still true even when our interim efficiency notion is adopted. However, when allocations are private information measurable, the adoption of our notion of interim private efficiency (Definition 5.3.2) guarantees that indeed any interim private efficient allocation is always CBIC (BIC). One may think of Corollary 6.2 as an improvement of that of Myerson (1979), in the sense that a stronger notion of interim efficiency with a simple condition (private information measurability) makes any interim private efficient allocation CBIC (BIC).

Example 6.1: Consider an economy with differential information with two agents, two goods, and two equally probable states, where utility functions, random initial endowments, and private information sets are given as follows:

$$u_1(\omega, x^1, x^2) = \sqrt{x^1 x^2}, \quad e_1 = ((10,0),(10,0)), \quad \mathscr{F}_1 = \{\{\omega_1, \omega_2\}\},$$
$$u_2(\omega, x^1, x^2) = \sqrt{x^1 x^2}, \quad e_2 = ((0,8),(0,10)), \quad \mathscr{F}_2 = \{\{\omega_1\}, \{\omega_2\}\}.$$

The allocation $x = (x_1, x_2)$ with

$$x_1 = ((5,4),(5,5)),$$
$$x_2 = ((5,4),(5,5))$$

is a strongly ex post efficient (ex post efficient, or HM ex post efficient) allocation. But it is neither ex ante nor interim efficient allocation because x_i is not $\mathscr{F}_i$-measurable for $i = 1, 2$. However, if we do not impose private information measurability on the allocations as in Myerson (1979), Holmström-Myerson (1983), and Palfrey-Srivastava (1987), this allocation is ex ante efficient, interim efficient, and ex post efficient. But, observe that it is not CBIC (BIC). Suppose that ω_2 is realized. Since

$$V_2(\omega_2, e_2 + (z \circ [\alpha_2, \alpha_1^*])_2) > V_2(\omega_2, x_2)$$

with $\alpha_2(E_2(\omega)) = \{\omega_1\}$ for every $\omega \in \Omega$, it is not CBIC (BIC). Therefore, this example shows that *an interim efficient allocation without private information measurability may not be* CBIC (BIC). This also illustrates that Bayesian incentive compatibility is incompatible with the ex post efficiency.

Proposition 6.12: An interim weak fine efficient allocation in $\mathscr{E}$ may not be CBIC.

Proof: Observe that the allocation x in Example 6.1 is also a weak fine efficient allocation. □

7 Are efficient and incentive compatible allocations individually rational?

Even though a mechanism is efficient, it cannot be achieved unless it is individually rational, otherwise someone may not be willing to trade. Therefore the individual rationality condition is a fundamental requirement for a mechanism. As with the efficiency notions, the individual rationality can be defined according to ex ante, interim, and ex post utility functions. In this section, we show that efficient allocations may not be individually rational.

Definition 7.1: An allocation $x \in \boldsymbol{A}$ is *interim individually rational* if for every $\omega \in \Omega, V_i(\omega, x_i) \geq V_i(\omega, e_i)$ holds for every $i \in I$.

An allocation $x \in \boldsymbol{A}$ is *ex ante individually rational* if the same condition holds for ex ante expected utility $\overline{V}_i$. An allocation $x \in \boldsymbol{A}^0$ is *ex post individually rational* if the same condition holds for ex post utility u_i and ex post feasible set $\boldsymbol{A}^0$.

We begin with a simple result for an economy with one good per state.

Proposition 7.1: The initial endowment is the unique interim individually rational allocation in $\bar{\mathscr{E}}$.

Proof: First of all, note that the initial endowment is interim individually rational. Suppose that a feasible allocation $x \neq e$ is individually rational. Then for every $\omega \in \Omega$ and every $i \in I$,

$$V_i(\omega, x_i) \geq V_i(\omega, e_i),$$

Since there is only one good and $x \neq e$, this implies that $x_i(\omega) \geq e_i(\omega)$ for every $\omega \in \Omega$ and $i \in I$, and $x_i(\omega^*) > e_i(\omega^*)$ for some $\omega^* \in \Omega$ and for some $i \in I$ by measurability. Thus, $\overline{e}(\omega^*) = \sum_{i \in I} x_i(\omega^*) > \sum_{i \in I} e_i(\omega^*) = \sum_{i \in I} \overline{e}(\omega^*)$, a contradiction. □

Proposition 7.2: An ex ante private efficient allocation in $\mathscr{E}$ may not be interim individually rational.

Proof: Consider an economy with differential information with three agents, one good, and three states ($i.e., \Omega = \{\omega_1, \omega_2, \omega_3\}$) with equal probability

($i.e.$, $\mu(\{\omega\}) = 1/3$ for every $\omega \in \Omega$) where utility functions, initial endowment, and private information sets are given as follows:

$$\begin{array}{lll} u_1(\omega, x) = \sqrt{x} & e_1 = (9,9,1) & \mathscr{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\} \\ u_2(\omega, x) = \sqrt{x} & e_2 = (9,1,9) & \mathscr{F}_2 = \{\{\omega_1, \omega_3\}, \{\omega_2\}\} \\ u_3(\omega, x) = \sqrt{x} & e_3 = (0,0,0) & \mathscr{F}_3 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}. \end{array}$$

It can be shown that the allocation $x = (x_1, x_2, x_3)$ is ex ante private efficient and ex ante individually rational where

$$x_1 = (8,8,2),\ x_2 = (8,2,8),\ x_3 = (2,0,0).$$

However, the initial endowment is the *unique* and interim individually rational allocation. □

Proposition 7.3: A CBIC (BIC) allocation in $\mathscr{E}$ may not be interim individually rational.

Proof: Consider an economy with differential information with two agents, one good, and two equally probable states, where utility functions, random initial endowments, and private information sets are given as follows:

$$\begin{array}{lll} u_1(\omega, x) = \sqrt{x}, & e_1 = (8,8), & \mathscr{F}_1 = \{\{\omega_1, \omega_2\}\}, \\ u_2(\omega, x) = \sqrt{x}, & e_2 = (1,1), & \mathscr{F}_2 = \{\{\omega_1\}, \{\omega_2\}\}. \end{array}$$

The allocation $x = (x_1, x_2)$ with

$$x_1 = (9,9),$$
$$x_2 = (0,0)$$

is a CBIC (BIC) allocation but is not interim individually rational.

8 On the existence of individually rational and efficient allocations

8.1 Existence of individually rational and efficient allocations

Before we state the results for the individually rational and efficient allocations, we need two preliminary lemmata. These concern the properties of the upper contour set and a selection theorem.

Lemma 8.1.1: Suppose that $u_i(\omega, \cdot)$ is upper semicontinuous and concave for every $\omega \in \Omega$. Define the correspondence $P_i : \Omega \times L_{X_i} \to 2^{L_{X_i}}$ by

$$P_i(\omega, x_i) = \{x_i' \in L_{X_i} : V_i(\omega, x_i') > V_i(\omega, x_i)\}.$$

Then for every $\omega \in \Omega$, $P_i(\omega, \cdot)$ is

(a) irreflexive, convex-valued, and
(b) it has weakly open lower sections[20].

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[20] Let X, Y be linear topological spaces. A correspondence $\Psi : X \to 2^X$ is said to be irreflexive if $x \notin \Psi(x)$ for every $x \in X$. A correspondence $\Psi : X \to 2^Y$ is said to have (weakly) open lower sections if for every $y \in Y$, the set $\Psi^{-1}(y) := \{x \in X : y \in \Psi(x)\}$ is (weakly) open in X.

Proof: (a) It follows from the concavity of $u_i(\omega, \cdot)$ that $V_i(\omega, \cdot)$ is concave as well and therefore the correspondence $P_i(\omega, \cdot)$ is convex-valued. It can be easily checked that $P_i(\omega, \cdot)$ is irreflexive, for if $x_i \in P_i(\omega, x_i)$ for some x_i, then $V_i(\omega, x_i) > V_i(\omega, x_i)$, a contradiction.
(b) Fix $\omega \in \Omega$. To show that $P_i(\omega, \cdot)$ has weakly open lower sections in L_{X_i}, define the correspondence $R_i : \Omega \times L_{X_i} \to 2^{L_{X_i}}$ by

$$R_i(\omega, x_i) = L_{X_i} \backslash P_i^{-1}(\omega, x_i) = \{x_i' \in L_{X_i} : V_i(\omega, x_i') \geq V_i(\omega, x_i)\}.$$

It suffices to show that $R_i(\omega, x_i)$ is weakly closed for every x_i. Fix x_i and take a net $\{x_i^\lambda\}$ such that x_i^λ converges weakly to x_i^* in L_{X_i} and $x_i^\lambda \in R_i(\omega, x_i)$. Since $x_i^\lambda \in R_i(\omega, x_i)$, it follows that $V_i(\omega, x_i^\lambda) \geq V_i(\omega, x_i)$. By Lemma 3.1.2, $V_i(\omega, \cdot)$ is weakly upper semicontinuous, *i.e.*, if x^λ converges weakly to x^*, we have $V_i(\omega, x_i^*) \geq \limsup V_i(\omega, x_i^\lambda)$.Notice that $\limsup V_i(\omega, x_i^\lambda) \geq V_i(\omega, x_i^\lambda)$. Therefore, $V_i(\omega, x_i^*) \geq V_i(\omega, x_i), i.e.,\ x_i^* \in R_i(\omega, x_i)$. Hence $R_i(\omega, x_i)$ is weakly closed and we can conclude that $P_i(\omega, \cdot)$ has weakly open lower sections in L_{X_i}. □

Lemma 8.1.2: If X be a paracompact Hausdorff space and Y be a topological space. Suppose that a correspondence $\Psi : X \to 2^Y$ is non-empty-valued, convex-valued, and having open lower sections. Then there exists a continuous function $f : X \to Y$ such that $f(x) \in \Psi(x)$ for every $x \in X$.

Proof: See Theorem 3.1 in Yannelis-Prabhakar (1983). □

For the theorem below we will assume that Ω in finite. This will simplify the proof.

Theorem 8.1.3: If $u_i(\omega, \cdot)$ is upper semicontinuous and concave for every $i \in I$ and every $\omega \in \Omega$, then an interim individually rational and weakly interim efficient allocation exists in $\mathscr{E}$.

Proof: Let $\boldsymbol{B}$ be the set of all interim individually rational allocations:

$$\boldsymbol{B} = \{x \in L_X : \forall \omega \in \Omega, V_i(\omega, x_i) \geq V_i(\omega, e_i), \forall i \in I\}.$$

Since $e \in \boldsymbol{B}$, $\boldsymbol{B}$ is nonempty. Since $V_i(\omega, \cdot)$ is weakly upper semicontinuous, $\boldsymbol{B}$ is a weakly closed subset of the order interval $[0, \bar{e}]^{|I|} = [0, \bar{e}] \times \cdots \times [0, \bar{e}]$, which is weakly compact (Cartwright's Theorem). This implies that $\boldsymbol{B}$ is also weakly compact.

Let $\bigwedge_{i \in I} \mathscr{F}_i = \{E^1, E^2, \cdots, E^k, \cdots, E^n\}$ be the common knowledge partition. Fix $E^k \in \bigwedge_{i \in I} \mathscr{F}_i$. Let us define

$$\begin{aligned} \boldsymbol{B}^k &= \{x \cdot \chi_{E^k} : x \in \boldsymbol{B}\}, \\ L_{X_i}^k &= \{x_i \cdot \chi_{E^k} : x_i \in L_{X_i}\}, \\ L_X^k &= \{x \cdot \chi_{E^k} : x \in L_X\}, \end{aligned}$$

Note that $\boldsymbol{B}^k$ is weakly compact. Define the correspondence $P_i^k : E^k \times L_{X_i}^k \to 2^{L_{X_i}^k}$ by

$$P_i^k(\omega, x_i \cdot \chi_{E^k}) = \{x_i' \cdot \chi_{E^k} \in L_{X_i}^k : V_i(\omega, x_i') > V_i(\omega, x_i)\}.$$

and define the correspondence $P^k : \boldsymbol{B}^k \to 2^{\boldsymbol{B}^k}$ by

$$P^k(x \cdot \chi_{E^k}) = \bigcap_{\omega \in E^k} \left[\prod_{i \in I} P_i^k(\omega, x_i \cdot \chi_{E^k}) \bigcap \boldsymbol{B}^k \right].$$

It follows from Lemma 8.1.1 that P^k is irreflexive, convex, and has weakly open lower sections in $\boldsymbol{B}^k$.

Now let $x \cdot \chi_{E^k} \in \boldsymbol{B}^k$ and suppose that there is $x' \cdot \chi_{E^k} \in L_X^k$ such that $x_i' \cdot \chi_{E^k} \in P_i(\omega, x_i \cdot \chi_{E^k})$ for every $\omega \in E^k$. Since $x \cdot \chi_{E^k}$ belongs to $\boldsymbol{B}^k$, so does $x' \cdot \chi_{E^k}$. Therefore, P^k is nonempty-valued. Hence, there is a weakly continuous function $f : B^k \to \boldsymbol{B}^k$ such that $f(x \cdot \chi_{E^k}) \in P^k(x \cdot \chi_{E^k})$ for every $x \cdot \chi_{E^k} \in \boldsymbol{B}^k$. By the Brouwer-Schauder-Tychonotf fixed point theorem, there exists a fixed point, i.e., $\bar{x} \cdot \chi_{E^k} \in f(\bar{x} \cdot \chi_{E^k}) \in P(\bar{x} \cdot \chi_{E^k})$, a contradiction to the irreflexivity of P^k. Therefore, there exists $x^k \cdot \chi_{E^k} \in \boldsymbol{B}^k$ such that $P^k(x^k \cdot \chi_{E^k}) = \emptyset$ for every $k = 1, \cdots, n$. Construct $x^* = \sum_{k=1}^n x^k \cdot \chi_{E^k}$. It is clear that x^* is interim individually rational. To show that it is weakly interim efficient, suppose otherwise. Then there is $x' \in \boldsymbol{A}$ such that for some common knowledge event $E^k \in \bigwedge_{i \in I} \mathscr{F}_i$, $V_i(\omega, x') > V_i(\omega, x^*)$ for every $\omega \in E^k$ and for every $i \in I$. This means $x_i' \cdot \chi_{E^k} \in P_i^k(\omega, x_i^k \cdot \chi_{E^k})$ for every $\omega \in E^k$ and for every $i \in I$. Since $x^k \cdot \chi_{E^k} \in \boldsymbol{B}^k$, it follows that $x' \cdot \chi_{E^k} \in \boldsymbol{B}^k$. This contradicts that $P^k(x^k \cdot \chi_{E^k}) = \emptyset$. □

Theorem 8.1.4: If the $u_i(\omega, \cdot)$ is upper semicontinuous and concave for every $i \in I$ and every $\omega \in \Omega$, then the set of ex ante individually rational and ex ante private efficient allocations of ε is nonempty.

Proof: Let $\boldsymbol{H}$ be the set of all ex ante individually rational allocations:

$$\boldsymbol{H} = \{x \in L_X : \overline{V}_i(x_i) \geq \overline{V}_i(e_i), \forall i \in I\}.$$

Since $e \in \boldsymbol{H}, \boldsymbol{H}$ is nonempty. Since $\overline{V}$ is weakly upper semicontinuous, $\boldsymbol{H}$ is a weakly closed subset of the order interval $[0, \overline{e}]^{|I|} = [0, \overline{e}] \times \cdots \times [0, \overline{e}]$, which is weakly compact (Cartwright's Theorem). This implies that $\boldsymbol{H}$ is also weakly compact. Define the correspondence $\bar{P}_i : L_{X_i} \to 2^{L_{X_i}}$ by

$$\bar{P}_i(x_i) = \{x_i' \in L_{X_i} : \overline{V}_i(x_i') > \overline{V}_i(x_i)\}.$$

and define the correspondence $\bar{P} : \boldsymbol{H} \to 2^{\boldsymbol{H}}$ by

$$\bar{P}(x) = \prod_{i \in I} \bar{P}_i(x_i) \bigcap \boldsymbol{H}.$$

In the same way as in Lemma 8.1.3, we can show that $\bar{P}$ is irreflexive, convex-valued, and it has weakly open lower sections in $\boldsymbol{H}$.

Now let x be an ex ante individually rational allocation. Suppose that it is not ex ante private efficient. Then there is an allocation $x' \in \boldsymbol{A}$ such that $x_i' \in \bar{P}_i(x_i)$ for every $i \in I$. Note that $x \in \boldsymbol{H}$ implies $x' \in \boldsymbol{H}$. It follows that $x' \in \bar{P}(x)$ and therefore, $\bar{P}$ is nonempty-valued. By Lemma 8.1.2, there is a weakly continuous function $f : \boldsymbol{H} \to \boldsymbol{H}$ such that $f(x) \in \bar{P}(x)$ for every $x \in \boldsymbol{H}$. By the Brouwer-Schauder-Tychonoff Theorem, there exists a fixed point $x^* = f(x^*) \in \bar{P}(x^*)$, a contradiction to the irreflexivity of $\bar{P}$. Hence we conclude that there exists an ex ante individually rational and ex ante efficient allocation. □

8.2 Nonexistence of individually rational and efficient allocations

Below we show that in well-behaved differential information economies, that is, where agents' utility functions are monotone, continuous, and concave, an interim fine efficient allocation may not exist.

Proposition 8.2.1: An interim fine efficient allocation may not exist in $\mathscr{E}$.

Proof: Consider an economy with differential information with two agents, two goods, and three equally probable states, where utility functions, random initial endowments, and private information sets are given as follows:

$$u_1(\omega, x^1, x^2) = \sqrt{x^1 x^2},\ e_1 = ((10,0),(10,0),(10,0)),\ \mathscr{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\},$$
$$u_2(\omega, x^1, x^2) = \sqrt{x^1 x^2},\ e_2 = ((10,0),(0,10),(0,10)),\ \mathscr{F}_2 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}.$$

Suppose that ω_1 is realized. Then there is no trade in that state. It follows that agent 1 will not trade at ω_2 and agent 2 will not trade ω_3, which implies that there is no trade at every state. The allocation $x = e$ is the unique feasible allocation. Consider a new allocation $x' \in \boldsymbol{A}(\bigvee_{i \in I} \mathscr{F}_i)$:

$$x'_1 = ((10,0),(5,5),(5,5))$$
$$x'_2 = ((10,0),(5,5),(5,5)).$$

Since $V_i(\omega_2, x'_i) > V_i(\omega_2, e_i)$ for $i = 1,2$, the initial endowment is not interim fine efficient. Hence there is no interim fine efficient allocation. □

Proposition 8.2.2: An interim individually rational and interim coarse efficient allocation need not exist in $\mathscr{E}$.

Proof: Consider an economy with differential information with two agents, two goods, and two equally probable states, where utility functions, random initial endowments, and private information sets are given as follows:

$$u_1(\omega, x) = \sqrt{x^1 x^2}, \quad e_1 = ((10,2),(10,2)), \quad \mathscr{F}_1 = \{\{\omega_1, \omega_2\}\},$$
$$u_2(\omega, x) = \sqrt{x^1 x^2}, \quad e_2 = ((2,10),(2,6)), \quad \mathscr{F}_2 = \{\{\omega_1\}, \{\omega_2\}\}.$$

The set of all interim coarse efficient allocation is

$$\{[((12,8),(12,8)),((0,4),(0,0))]; [((0,0),(0,0)),((12,12),(12,8))]\}.$$

Hence, no interim coarse efficient allocation is interim individually rational.

Corollary 8.2.3: An interim individually rational and interim private efficient allocation need not exist in $\mathscr{E}$.

Proof: Since the interim private efficiency implies the interim coarse efficiency [Proposition 5.4.2 (b)], the claim follows from Proposition 8.2.2. □

8.3 Compactness of the set of individually rational and efficient allocations

In this section, we show that the set of interim individually rational and interim private efficient allocations is weakly compact.

Theorem 8.3.1: If $u_i(\omega, \cdot)$ is upper semicontinuous for every $i \in I$ and every $\omega \in \Omega$, the set of interim individually rational and interim private efficient allocations of $\mathscr{E}$ is weakly compact.

Proof: Define the correspondence $P_i : \Omega \times L_{X_i} \to 2^{L_{X_i}}$ by

$$P_i(\omega, x_i) = \{x'_i \in L_{X_i} : V_i(\omega, x'_i) > V_i(\omega, x_i)\}.$$

and define the correspondence $P : \Omega \times \boldsymbol{B} \to 2^{L_X}$ by

$$P(\omega, x) = \prod_{i \in I} P_i(\omega, x_i). \tag{8.3.1}$$

It follows from Lemma 8.1.1 that $P(\omega, \cdot)$ is irreflexive, and has weakly open lower sections in $\boldsymbol{B}$ for every $\omega \in \Omega$. Let M be the set of interim individually rational and interim private efficient allocations. Formally,

$$M = \{x \in \boldsymbol{B} : \forall \omega, P(\omega, x) = \emptyset\}$$

Then it follows that

$$\begin{aligned}
\boldsymbol{B} \backslash M &= \{x \in \boldsymbol{B} : \exists \omega \in \Omega, P(\omega, x) \neq \emptyset\} \\
&= \{x \in \boldsymbol{B} : \exists \omega \in \Omega \text{ and } \exists y \in P(\omega, x)\} \\
&= \{x \in \boldsymbol{B} : \exists \omega \in \Omega \text{ and } \exists y \in L_X \text{ such that } x \in P^{-1}(\omega, y)\} \\
&= \bigcup_{\omega \in \Omega} \bigcup_{y \in L_X} P^{-1}(\omega, y)
\end{aligned}$$

Since $P(\omega, \cdot)$ has weakly open lower sections, $\boldsymbol{B} \backslash M$ is weakly open. Hence M is a weakly closed subset of the weakly compact set B and therefore we can conclude that M is weakly compact. □

Notice that if $u_i(\omega, \cdot)$ is affine (a rather strong assumption which rules out risk aversion), the set M can be shown to be convex. This is parallel to the results of Myerson (1979) and Holmström-Myerson (1983) who show that if $u_i(\omega, \cdot)$ is linear, the set M is convex.

Theorem 8.3.2: If the $u_i(\omega, \cdot)$ is upper semicontinuous for every $i \in I$ and every $\omega \in \Omega$, then the set of ex ante individually rational and ex ante private efficient allocations of $\mathscr{E}$ is weakly compact.

Proof: Define the correspondence $\bar{P}_i : L_{X_i} \to 2^{L_{X_i}}$ by

$$\bar{P}_i(x_i) = \{x'_i \in L_{X_i} : \overline{V}_i(x'_i) > \overline{V}_i(x_i)\}.$$

and define the correspondence $\bar{P} : \boldsymbol{H} \to 2^{L_X}$ by

$$\bar{P}(x) = \prod_{i \in I} \bar{P}_i(x_i). \tag{8.3.2}$$

Define $\boldsymbol{H}$ as in Theorem 8.1.4. Then $\boldsymbol{H}$ is nonempty and weakly compact, and $\bar{P}(\cdot)$ is irreflexive and has weakly open lower sections in $\boldsymbol{H}$. Let M^a be the set of ex ante individually rational and ex ante private efficient allocations. Formally,

$$M^a = \{x \in \boldsymbol{H} : \bar{P}(x) = \emptyset\}$$

Then it follows that

$$\begin{aligned} \boldsymbol{H} \setminus M^a &= \{x \in \boldsymbol{H} : \bar{P}(x) \neq \emptyset\} \\ &= \{x \in \boldsymbol{H} : \exists y \in \bar{P}(x)\} \\ &= \{x \in \boldsymbol{H} : \exists y \in L_X \text{ such that } x \in \bar{P}^{-1}(\omega, y)\} \\ &= \bigcup_{y \in L_X} \bar{P}^{-1}(y) \end{aligned}$$

Since $\bar{P}$ has weakly open lower sections, $\boldsymbol{H} \setminus M^a$ is weakly open. Hence M^a is a weakly closed subset of the weakly compact set $\boldsymbol{H}$ and therefore we can conclude that M^a is weakly compact. □

Theorem 8.3.3: If the $u_i(\omega, \cdot)$ is upper semicontinuous for every $i \in I$ and every $\omega \in \Omega$, then the set of ex post individually rational and ex post efficient allocations of $\mathscr{E}$ is nonempty and weakly compact.

Proof: One can proceed in a similar way as in Theorem 8.3.1. □

9 Incentive efficiency

As we saw in Example 6.1, the private information measurability of allocations is the key assumption to obtain our results. That is, without it, an interim efficient allocation may be not Bayesian incentive compatible. We now propose the concept of incentive efficiency. This notion is defined as before but no measurability conditions are imposed. Hence, we disregard the private information measurability assumption and define the concept of incentive efficiency. We show that an incentive efficient allocation exists and that the set of incentive efficient allocations is weakly compact. This kind of approach is not the first one in the literature. Holmström-Myerson (1983) show that an efficient (HM interim efficient) allocation may be not Bayesian incentive compatible and propose the concept of incentive efficiency as an appropriate efficiency concept in incomplete information environment. However, we have a different setting, *i.e.*, a differential information economy, rather than the Harsanyi model. Moreover, we allow for a continuum of states and commodities as well. We define below our incentive efficiency notion. Note that we keep all the definitions of efficiency and incentive compatibility as before except the imposition of private information measurability on allocations. An allocation $x = e + z \in \boldsymbol{A}^0$ is *incentive compatible* if for every $\omega \in \Omega$, every $i \in I$, and every deception $\alpha_i : \mathscr{F}_i \to \mathscr{F}_i$ such that

$(\alpha_i, \alpha^*_{-i})$ is compatible with F and $V_i(\omega, x_i) \geq V_i(\omega, e_i + (z \circ [\alpha_i, \alpha^*_{-i}])_i)$ where $e + z \circ [\alpha_i, \alpha^*_{-i}] \in \boldsymbol{A}^0$. Let us define

$$D_i = \{\alpha_i | \alpha_i : \mathscr{F}_i \to \mathscr{F}_i\}$$

and let

$$C_i(\omega, \alpha_i) = \{x \in \boldsymbol{A}^0 : V_i(\omega, x_i) \geq V_i(\omega, e_i + (z \circ [\alpha_i, \alpha^*_{-i}])_i) \text{ with } e + z \circ [\alpha_i, \alpha^*_{-i}] \in \boldsymbol{A}^0\}.$$

Then the set of all incentive compatible allocations is given by

$$\boldsymbol{C} = \bigcap_{\omega \in \Omega} \bigcap_{i \in I} \bigcap_{\alpha_i \in D_i} C_i(\omega, \alpha_i).$$

Definition 9.1: An allocation $x = e + z \in \boldsymbol{C}$ is *incentive efficient* if there is no $x' \in \boldsymbol{C}$ such that for some $\omega \in \Omega$, $V_i(\omega, x'_i) > V_i(\omega, x_i)$ for every $i \in I$.

It should be noted that our interim efficiency (Definition 5.3.2) is not comparable to the incentive efficiency without private information measurability because the set of feasible allocations with the private information measurability assumption is smaller but it is more difficult for the grand coalition to block them with the private information measurability assumption. Hence, neither set contains the other one. As in Myerson (1979), for the existence of incentive efficient allocations, risk neutrality is required as the theorem below indicates (compare this result with the existence results of Section 8 where risk aversion is allowed).

Theorem 9.2: If $u_i(\omega, \cdot)$ is continuous and affine $\omega \in \Omega$ and $i \in I$, there exists an incentive efficient allocation in $\mathscr{E}$.

Proof: Since $e \in \boldsymbol{C}$, $\boldsymbol{C}$ is nonempty. Since $V_i(\omega, \cdot)$ is weakly continuous by Lemma 3.1.3, it follows that $\boldsymbol{C}$ is a weakly closed subset of the weakly compact set $[0, \bar{e}]^{|I|} = [0, \bar{e}] \times [0, \bar{e}] \times \cdots \times [0, \bar{e}]$. This implies that $\boldsymbol{C}$ is weakly compact.

Fix $\omega \in \Omega$ and consider the maximization problem:

$$\max_{x \in \boldsymbol{C}} \sum_{i \in I} \lambda_i(\omega) u_i(\omega, x_i(\omega)),$$

where for all $i \in I$, $\lambda_i : \Omega \to \boldsymbol{R}_+$ and $\lambda = (\lambda_i)_{i \in I} \neq 0$. Since C is nonempty and weakly compact, and the maximand is weakly continuous in x, there is a solution x^ω. Then x^* with $x^*(\omega) = x^\omega(\omega)$ is an incentive efficient allocation. □

Theorem 9.3: If $u_i(\omega, \cdot)$ is continuous and affine for every $\omega \in \Omega$ and for every $i \in I$, the set of incentive efficient allocations of $\mathscr{E}$ is weakly compact.

Proof: One can proceed in a similar way as in Theorem 8.3.1. □

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