Econ Theory (2009) 38:385–397
DOI 10.1007/s00199-007-0288-6

SYMPOSIUM

Bayesian–Walrasian equilibria: beyond the rational expectations equilibrium

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Received: 18 April 2007 / Accepted: 5 September 2007 / Published online: 2 October 2007

Abstract In general rational expectations equilibrium (REE), as introduced in Radner (Econometrica 47:655–678, 1978) in an Arrow–Debreu–McKenzie setting with uncertainty, does not exist. Moreover, it fails to be fully Pareto optimal and incentive compatible and is also not implementable as a perfect Bayesian equilibrium of an extensive form game (Glycopantis et al. in Econ Theory 26:765–791, 2005). The lack of all the above properties is mainly due to the fact that the agents are supposed to predict the equilibrium market clearing price (as agent's expected maximized utility is conditioned on the information that equilibrium prices reveal), which leads inevitably to the presumption that agents know all the primitives in the economy, i.e., random initial endowments, random utility functions and private information sets. To get around this problematic equilibrium notion, we introduce a new concept called *Bayesian–Walrasian equilibrium* (BWE) which has Bayesian features. In particular, agents try to predict the market-clearing prices using Bayesian updating and evaluate their consumption in terms of Bayesian price estimates, which are different for each individual. In this framework agents maximize expected utility conditioned on their own private information about the state of nature, subject to a Bayesian estimated budget constraint. Market clearing is not an intrinsic part of the definition of BWE. However, both in the case of perfect foresight and in the case of symmetric information

We wish to thank an associate editor and two knowledgeable referees for useful comments and suggestions.

This work was done in the Spring of 2005, when EJB was a visiting professor at the University of Illinois.

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BWE leads to a statewise market clearing; it then becomes an ex post Walrasian equilibrium allocation. This new BWE exists under standard assumptions, in contrast to the REE. In particular, we show that our new BWE exists in the well-known example in Kreps (J Econ Theory 14:32–43, 1977), where REE fails to exist.

Keywords Bayesian Walrasian expectations equilibrium · Rational expectations equilibrium

JEL Classification D51 · D82 · C71

1 Introduction

The deterministic Walrasian equilibrium (WE) concept captures the idea of exchange or contracts or trades of goods under complete information. Since this concept exists (under reasonable assumptions), is Pareto optimal and is implementable as Nash equilibrium of a game, one can automatically infer that the WE contracts lead to nice outcomes. In reality however, most contracts are made under uncertainty. To this end three main extensions of the deterministic WE notion were made to incorporate uncertainty. The first one is due to Arrow and Debreu (see for example Chap. 7 of the classical treatise Debreu 1959). These authors noticed that once agents' preferences and initial endowments are random (depend on the states of nature of the world) the standard existence and optimality theorems for the deterministic WE continue to hold. This is the so-called "state contingent model" or complete markets model (same number of markets as states of nature) and everything works like the deterministic model, i.e., the existence, optimality and implementation results continue to hold. However, this model does not allow for asymmetric information, because it supposes the state of nature to be public knowledge.

The more complicated situation with differentiated information was studied by Radner (1968). In this second extension of the WE, in addition to random preferences and initial endowments, he allowed each agent to have a private information set (which is a partition of the exogenously given states of nature of the world). In this model agents maximize ex ante expected utility subject to an ex ante budget constraint. However, all trades (allocations) made are measurable with respect to the private information of each agent and thus asymmetric information was explicitly introduced by Radner.[1] This model captures the idea of contracts made in an ex ante stage under asymmetric information. The corresponding notion is called Walrasian expectations equilibrium (WEE), and it exists under reasonable assumptions, as shown by Radner. In this model one worries about the incentives that individuals have to misreport their private information. However, it is known (see for example, Herves-Beloso et al. 2005 or Podczeck and Yannelis 2005, among others) that the WEE is coalitional Bayesian incentive compatible, it is Pareto optimal (as it belongs to the private core) and it is also implementable as a perfect Bayesian equilibrium of an extensive form game (see Glycopantis and Yannelis 2005).

[1] For an interpretation of the private information measurability assumption and its consequences, see Podczeck and Yannelis (2005).

The third extension was also made by Radner (1972), who introduced the concept of a rational expectations equilibrium REE. This is an interim notion and agents maximize conditional expected utility (interim expected utility) based not only on their own private information, but also on the information that the equilibrium prices have generated. The resulting allocation clears the market for every state of nature. Since agents maximize their interim expected utility conditioned on the information that the equilibrium prices have generated, this leads inevitably to the presumption that each agent knows precisely all the primitives in the economy (i.e., random preferences, random initial endowments, private information sets and priors of all other agents). This is somewhat difficult to justify, since only rather omniscient agents could act accordingly. Moreover, Radner could only prove the existence of such REE in a generic sense. Hence it should not come as a surprise that in simple, well-behaved economies the REE may not exist (Kreps 1977). Moreover, REE may not be Pareto optimal, may not be incentive compatible and may not be implementable as a perfect Bayesian equilibrium of an extensive form game (Glycopantis and Yannelis 2005). In other words, the resulting contracts may not have any of the desirable properties that we would like any reasonable contract to have. This suggests that a new notion is called for, which is free of the undesirable properties of the REE.

In what we see as a first stage step towards this goal (in our future studies learning effects will also be incorporated), this paper introduces a new equilibrium notion, called Bayesian–Walrasian equilibrium (BWE). In particular, we do not assume that agents maximize interim expected utility based on the information that the equilibrium prices generate. To the contrary, we assume that, in what may be seen as a first, provisional stage of the trading process, agents form price estimates based on their own private information; in terms of those prices they can formulate *estimated budget sets*. Based on his/her own private information, each agent then maximizes interim expected utility, constrained by his/her own estimated budget set. As a consequence of the imprecision due to price estimation, the resulting equilibrium allocation may not clear the markets for every state of nature, unless information is symmetric or agents perfectly forecast the equilibrium price. In general, the properties of BWE only imply that the feasibility-market clearing condition is in terms of an expectation. It turns out that, in contrast to the REE, the BWE exists under the same standard assumptions that guarantee the existence of the deterministic WE or the ex ante personalized Walrasian expectations equilibrium in Aliprantis et al. (2001). In particular, this implies that a BWE exists in the above-mentioned counterexample to existence of REE, as given in (Kreps 1977).

This paper is organized as follows. In Sect. 2 we present the new notion. In Sect. 3 we revisit the example by Kreps (1977) an show explicitly that a BWE exists in this example. Section 4 contains the main existence result. All proofs are collected in Sect. 5. In Sect. 6 we conclude with some remarks and open questions.

2 Bayesian–Walrasian equilibria

We consider the following model of a differential information exchange economy $\mathcal{E}$, describing an economy with agents numbered $1, \ldots, N$ who have private information. We shall write $I := \{1, 2, \ldots, N\}$ for the set of all agents. Let Ω be a finite space of *states of nature*. Let $\mathcal{F}$ be an algebra[2] on Ω and let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$. Rather than relabeling the atoms of Ω, we shall suppose that $\mathcal{F}$ contains all singletons $\{\omega\}$ (i.e., $\mathcal{F}$ equals 2^{Ω}, the collection of all subsets of Ω and its only atoms are the singletons). Thus, for every state of nature $\omega \in \Omega$ the probability $\mathbb{P}(\omega)$ is well-defined. For every $i \in I$ we let $\mathcal{F}_i \subset \mathcal{F}$ be agent i's *informational algebra*. From now on, rather than restricting ourselves to the restrictions of the algebras $\mathcal{F}_i$ and $\mathcal{F}$ to the $\mathcal{F}$-measurable set $\Omega' := \{\omega \in \Omega : \mathbb{P}(\omega) > 0\}$,[3] we shall suppose without loss of generality that $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$. For every $\omega \in \Omega$ there is a unique atom of $\mathcal{F}_i$ that contains ω; we denote it by $F_i(\omega)$. Hence, upon the realization of $\omega \in \Omega$, which agent i perceives through $F_i(\omega)$, she forms the conditional probability given by

$$\mathbb{P}_i(\omega' \mid \omega) = \begin{cases} \frac{\mathbb{P}(\omega')}{\mathbb{P}(F_i(\omega))} & \text{if } \omega' \in F_i(\omega), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $X_i : \Omega \to 2^{\mathbb{R}^d_+}$ be agent i's *random consumption set*, let $u_i : \Omega \times \mathbb{R}^d_+ \to \mathbb{R}$ be her *random utility function* and let $e_i : \Omega \to \mathbb{R}^d_+$ be her *random initial endowment*. By $L_{X_i}(\mathcal{F}_i)$ we denote the set of all information-compatible *allocations* for agent i, i.e., the set of all functions $f_i : \Omega \to \mathbb{R}^d_+$ that are $\mathcal{F}_i$-measurable and such that $f_i(\omega) \in X_i(\omega)$ for every ω. We suppose that e_i belongs to $L_{X_i}(\mathcal{F}_i)$ for every $i \in I$.

Agent i's *interim expected utility* is the function $v_i : \Omega \times \mathbb{R}^d_+ \to \mathbb{R}$, defined by

$$v_i(\omega, x_i) := \mathbb{E}(u_i \mid \mathcal{F}_i)(\omega, x_i) := \sum_{\omega'} u_i(\omega', x_i)\mathbb{P}_i(\omega' \mid \omega).$$

As usual, we suppose $\mathcal{F} = \bigvee_{i=1}^N \mathcal{F}_i$. Let $\Delta := \{\lambda \in \mathbb{R}^d_+ : \sum_{j=1}^d \lambda_j = 1\}$ be the unit simplex, consisting of all normalized price vectors (this normalization is innocuous: multiplication of the prices by a common scalar does not affect the estimated budget sets to be defined below). Given a random price vector $p : \Omega \to \Delta$ (i.e., p is measurable with respect to $\mathcal{F} = 2^{\Omega}$), every agent i adopts following the conditional expectation:

$$\hat{p}_i(\omega) := \sum_{\omega' \in \Omega} p(\omega')\mathbb{P}_i(\omega' \mid \omega). \tag{2.2}$$

In other words, $\hat{p}_i(\omega)$ is agent i's *Bayesian price estimate* of the random price vector p, given that the state ω has been realized.

Using this natural estimate for the price, given her informational algebra, agent i forms the following *estimated budget set*:

$$\hat{B}_i(\omega, p) := \{x_i \in X_i(\omega) : \hat{p}_i(\omega) \cdot x_i \leq \hat{p}_i(\omega) \cdot e_i(\omega)\}.$$

[2] Because our space Ω is finite, the notions of algebra and σ-algebra coincide. Any algebra $\mathcal{G}$ on Ω is automatically generated by a partition, namely the partition consisting of all its atoms Neveu 1964, Proposition I.2.1. Recall that $G \in \mathcal{G}$ is an *atom* of $\mathcal{G}$ if $G' \subset G$ implies either $G' = \emptyset$ or $G' = G$ for every $G' \in \mathcal{G}$.

[3] The restriction of an algebra $\mathcal{G}$ on Ω to $\Omega' \in \mathcal{G}$ is the algebra defined by $\mathcal{G}' := \{G \in \mathcal{G} : G \subset \Omega'\}$.

Definition 2.1 A *Bayesian–Walrasian equilibrium* (BWE) of the differential information exchange economy $\mathcal{E}$ is a pair (p^*, f^*) such that

(i) p^* is a random price vector $p^* : \Omega \to \Delta$,
(ii) $f^* = (f_i^*)_{i\in I} \in \Pi_{i\in I} L_{X_i}(\mathcal{F}_i)$ is an allocation,
(iii) $f_i^*(\omega) \in \operatorname{argmax}_{x_i \in \hat{B}_i(\omega, p^*)} v_i(\omega, x_i)$ for every $\omega \in \Omega$ and every $i \in I$,
(iv) $p^*(\omega) \cdot \sum_{i\in I}(f_i^*(\omega) - e_i(\omega)) = \max_{1\leq j\leq d} \sum_{i\in I}(f_i^*(\omega) - e_i(\omega))_j$ for every $\omega \in \Omega$.

Observe that (iii) states that for every state of nature agent i maximizes her utility over her *estimated* budget set. It is easy to see that (iv) has the following equivalent and alternative form (iv)$^{\text{alt}}$, which can be useful for computations:

(iv)$^{\text{alt}}$ $(p^*(\omega))_k > 0$ implies $\sum_{i\in I}(f_i^*(\omega) - e_i(\omega))_k = \max_{1\leq j\leq d} \sum_{i\in I}(f_i^*(\omega) - e_i(\omega))_j$ for every $\omega \in \Omega$ and every $k = 1, \dots, d$.

Both (iv) and (iv)$^{\text{alt}}$ form an unusual substitute for the classical feasibility property of Walrasian equilibria. This must be considered the price that has to be paid when one wishes to avoid the unrealistic hypotheses surrounding REE, but below we shall also give special conditions under which feasibility becomes exact. We would like to stress that, as a notion, BWE must be considered exact. This should be contrasted to the literature on approximate REE, where the real issue is not the existence of universal approximate (or epsilon-approximate) REE solutions, but the question of whether the error (i.e., the epsilon) is computable and, if so, under which conditions. It appears to us that nothing is known about this problem. To put it differently, we know of no analogue for REE of the well-known state of affairs surrounding deterministic Walrasian equilibria, where, an increasing number of agents may compensate for convexity deficiencies of the preferences. At any rate, general truths of such a nature are out of the question for REE (the private information of a single agent may never become negligible when the number of agents goes to infinity; thus the approximate REE will not become an ex post Walrasian equilibrium allocation).

However, as the following result shows, in at least two different cases a BWE has feasibility. The proof of this result is deferred to Subsect. 5.1.

Proposition 2.1 *If (p^*, f^*) is a BWE as in Definition 2.1, then for every $F \in \cap_{i\in I}\mathcal{F}_i$*

$$\sum_{\omega\in F} \max_{1\leq j\leq d} \sum_{i\in I} (f_i^*(\omega) - e_i(\omega))_j \mathbb{P}(\omega) \leq 0.$$

Moreover, (free disposal) feasibility

$$\sum_{i\in I}(f_i^*(\omega) - e_i(\omega)) \leq 0 \textit{ for every } \omega \in \Omega$$

holds in each of the following cases:

(a) $\hat{p^*}_i = p^*$ *for all $i \in I$ (i.e., perfect forecasting of p^* by all agents).*
(b) $\mathcal{F}_i = \mathcal{F}_N$ *for all $i \in I$ (i.e., symmetric information).*

Moreover, if in addition $u_i(\omega, \cdot)$ is strongly monotonic on $X_i(\omega) := \mathbb{R}^d_+$ for every $i \in I$ and every $\omega \in \Omega$, then in case (a) the above feasibility condition sharpens into

$$\sum_{i \in I} (f_i^*(\omega) - e_i(\omega)) = 0 \quad \text{for every } \omega \in \Omega.$$

The same is true for case (b), provided that $p^*(\omega) \in \mathbb{R}^d_{++}$ for every $\omega \in \Omega$.

Let us compare this new BWE notion with the classical notion of a rational expectations equilibrium (REE) as in Radner (1979), etc. For $\omega \in \Omega$, $\lambda \in \Delta$ and $i \in I$ let

$$B_i(\omega, \lambda) := \{x_i \in X_i : \lambda \cdot x_i \leq \lambda \cdot e_i(\omega)\}.$$

This is the usual budget set for agent i under the state ω and under the price $\lambda \in \Delta$.

Definition 2.2 A *rational expectations equilibrium* is a pair (p^*, f^*) such that

(i) p^* is a random price vector $p^* : \Omega \to \Delta$.
(ii) $f^* = (f_i^*)_{i \in I} \in \Pi_{i \in I} L_{X_i}(\mathcal{G}_i^*)$ is an allocation, where $\mathcal{G}_i^* := \mathcal{F}_i \vee \sigma(p^*)$ with $\sigma(p^*) \subset \mathcal{F}$ denoting the (σ-)algebra generated by p^*.
(iii) $f_i^*(\omega) \in \text{argmax}_{x_i \in B_i(\omega, p^*(\omega))} \mathbb{E}(u_i \mid \mathcal{G}_i^*)(\omega, x_i)$ for every $\omega \in \Omega$ and $i \in I$.
(iv) $\sum_{i \in I} f_i^*(\omega) = \sum_{i \in I} e_i(\omega)$ for every $\omega \in \Omega$.

Such a REE (p^*, f^*) is said to be *fully revealing* if $\mathcal{G}_i^* := \mathcal{F}_i \vee \sigma(p^*) = \vee_{j \in I} \mathcal{F}_j$ for every $i \in I$.

Observe that in the case of a fully revealing REE one has both perfect forecasting by all agents and symmetric information. It is well-known that rational expectations equilibria need not exist. In contrast, it turns out that Bayesian–Walrasian equilibria exist under fairly standard conditions: see the next section and Sect. 4.

3 The Kreps example revisited

To highlight this state of affairs, we shall now consider a well-known example of Kreps (1977), which, does not allow a rational expectations equilibrium in the sense of Definition 2.2. However, below we shall concretely demonstrate that in that same example a unique Bayesian–Walrasian equilibrium exists.

Example 3.1 Let $N = 2$, $d = 2$ (2 agents, 2 goods) and let $\Omega = \{\omega_1, \omega_2\}$, with each state being considered equally probable by each agent, i.e., $\mathbb{P}(\omega_i) = 1/2$. Suppose that $\mathcal{F}_1 = 2^\Omega$ and that $\mathcal{F}_2 = \{\emptyset, \Omega\}$. Note already that this causes $A_1 := \Omega$ to be the only nonempty atom of $\mathcal{F}_1 \cap \mathcal{F}_2$. Here agent 1 uses $\mathbb{P}_1(\omega' \mid \omega) := 1$ if $\omega' = \omega$ and $\mathbb{P}_1(\omega' \mid \omega) := 0$ if $\omega' \neq \omega$. Also, agent 2 uses the uniform prior $\mathbb{P}_2(\omega' \mid \omega) = 1/2$. In Kreps' example the initial endowments are given by $e_1(\omega_j) = e_2(\omega_j) = (3/2, 3/2)$, $j = 1, 2$, and the utility functions are

$$\begin{aligned} u_1(\omega_1, \xi) &: = \log \xi_1 + \xi_2, \; u_1(\omega_2, \xi) := 2 \log \xi_1 + \xi_2, \\ u_2(\omega_1, \xi) &:= 2 \log \xi_1 + \xi_2, \; u_2(\omega_2, \xi) := \log \xi_1 + \xi_2, \end{aligned}$$

with $\xi := (\xi_1, \xi_2)$. Then we have of course $v_1 = u_1$ and for $j = 1, 2$

$$v_2(\omega_j, \xi) = \frac{1}{2}u_2(\omega_1, \xi) + \frac{1}{2}u_2(\omega_2, \xi) = \frac{3}{2}\log \xi_1 + \xi_2.$$

Standard arguments show the following for the optimal consumption bundles, chosen by the agents according to equilibrium condition (iii) in Definition 2.1. For agent 1 we have in state ω_1,

$$f_1^*(\omega_1) = \left(\frac{(p^*(\omega_1))_2}{(p^*(\omega_1))_1}, \frac{1.5 - (p^*(\omega_1))_2}{(p^*(\omega_1))_2}\right) = \left(\frac{1}{\alpha} - 1, \frac{3}{2(1-\alpha)} - 1\right),$$

where we abbreviate by writing $\alpha := (p^*(\omega_1))_1$ and substituting $(p^*(\omega_1))_2 = 1 - \alpha$. Simplifying in the same way with $\beta := (p^*(\omega_2))_1$, we have for agent 1 in state ω_2 that

$$f_1^*(\omega_2) = \begin{cases} \left(\frac{2}{\beta} - 2, \frac{3}{2(1-\beta)} - 2\right) & \text{if } \beta \geq \frac{1}{4}, \\ \left(\frac{3}{2\beta}, 0\right) & \text{if } \beta < \frac{1}{4}. \end{cases}$$

For agent 2 we have for $j = 1, 2$, in the same notation after simplifying:

$$f_2^*(\omega_j) = \left(\frac{3}{\alpha + \beta} - \frac{3}{2}, \frac{3}{2 - \alpha - \beta} - \frac{3}{2}\right).$$

It is easy to verify that neither α nor β can be equal to 0 or 1. Thus, by $(\text{iv})^{\text{alt}}$ we conclude that α and β satisfy

$$\frac{1}{\alpha} + \frac{3}{\alpha + \beta} = \frac{3}{2(1-\alpha)} + \frac{3}{2 - \alpha - \beta}, \quad \frac{2}{\beta} + \frac{3}{\alpha + \beta} = \frac{3}{2(1-\beta)} + \frac{3}{2 - \alpha - \beta}.$$

The unique solution of this system is $\alpha = 0.4082917393$ and $\beta = 0.5774105211$.[4] The corresponding excess demands are -0.007255532 in state ω_1 and $+0.007255532$ in state ω_2. The unique BWE pair (p^*, f^*) that corresponds to these values of α and β is given by

$$p^*(\omega_1) = (0.408, 0.592), \;\; p^*(\omega_2) = (0.577, 0.423),$$
$$f_1^*(\omega_1) = (1.449, 1.535), \;\; f_1^*(\omega_2) = (1.464, 1.550), \;\; f_2^*(\omega_j) = (1.544, 1.458).$$

Let us observe that in the above example both WEE (Radner 1968) and private core (Yannelis 1991) exist and coincide with the initial endowment. This underlines the fact that BWE is a different notion altogether (also, the initial endowments always belong to the estimated budget set, so it is not surprising that the corresponding BWE-expected utility values are higher).

4 Analytically, calculations in MAPLE show α to be a real root of $408z^5 - 963z^4 + 594z^3 - z^2 - 64z + 8$, with $\beta = \frac{1}{15941}(-10822\alpha + 26640 - 39576\alpha^4 + 148899\alpha^3 - 132282\alpha^2)$.

4 Main results: existence

In this section we state and prove our main existence result, for which we enlist the following assumptions.

Assumption 4.1 For every $i \in I$ and $\omega \in \Omega$ the set $X_i(\omega) \subset \mathbb{R}^d_+$ is closed, convex and nonempty for every $\omega \in \Omega$.

Assumption 4.2 For every $i \in I$ and $\omega \in \Omega$ the initial endowment $e_i(\omega)$ belongs to the interior of $X_i(\omega)$.

Assumption 4.3 For every $i \in I$ and $\omega \in \Omega$ the function $u_i(\omega, \cdot)$ is continuous and concave on $X_i(\omega)$.

Theorem 4.1 *Under Assumptions 4.1, 4.2, 4.3 there exists a Bayesian–Walrasian equilibrium pair for the differential information economy $\mathcal{E}$.*

5 Proofs

5.1 Proof of Proposition 2.1

The first part of Proposition 2.1 follows by the next lemma, which states a well-known property of conditional expectations (Neveu 1964).

Lemma 5.1 (i) *For every $i \in I$ and every $\mathcal{F}$-measurable function $\phi : \Omega \to \mathbb{R}$*

$$\sum_{\omega \in F_i} \phi(\omega)\mathbb{P}(\omega) = \sum_{\omega \in F_i} \mathbb{E}(\phi \mid \mathcal{F}_i)(\omega)\mathbb{P}(\omega) \quad \textit{for every } F_i \in \mathcal{F}_i.$$

(ii) *For every $i \in I$ and every $\mathcal{F}$-measurable function $\phi : \Omega \to \mathbb{R}$*

$$\sum_{\omega \in \Omega} \phi(\omega)\psi(\omega)\mathbb{P}(\omega) = \sum_{\omega \in \Omega} \psi(\omega)\mathbb{E}(\phi \mid \mathcal{F}_i)(\omega)\mathbb{P}(\omega)$$
$$\textit{for every } \mathcal{F}_i\textit{-measurable function } \psi : \Omega \to \mathbb{R}.$$

In the context of this paper the proofs are elementary. Because F_i is the union of all $\mathcal{F}_i$-atoms $F_i(\omega)$, $\omega \in F_i$, it is enough to prove the identity in part i when ϕ is of the form $\phi = 1_A$, with $A \in \mathcal{F}$. Then $\mathbb{E}(\phi \mid \mathcal{F}_i)(\omega) = \mathbb{P}(A \cap F_i(\omega))/\mathbb{P}(F_i(\omega))$ and the identity follows from the fact that the $\mathcal{F}_i$-atoms $F_i(\omega)$ partition the set F_i. Part ii then follows from part i, because ψ is actually a step function.

To prove the first part of Proposition 2.1, we apply this lemma for each i to $\hat{p}_i^* := \mathbb{E}(p^* \mid \mathcal{F}_i)$ and the $\mathcal{F}_i$-measurable function $(f_i^* - e_i)1_F$. This gives

$$\sum_{\omega \in F} p^*(\omega) \cdot (f_i^*(\omega) - e_i(\omega))\mathbb{P}(\omega) = \sum_{\omega \in F} \hat{p}_i^*(\omega) \cdot (f_i^*(\omega) - e_i(\omega))\mathbb{P}(\omega), \tag{5.1}$$

and by Definition 2.1 (iii) the right hand side is nonpositive. By summation over i and the use of of Definition 2.1 (iv) in the left hand side of (5.1), the result follows.

To prove the second part of Proposition 2.1, simply notice that in case (a) Definition 2.1 (iv) states that for every $\omega \in \Omega$,

$$\max_{1\le j\le d} \sum_{i\in I} (f_i^*(\omega) - e_i(\omega))_j = \sum_{i\in I} \hat{p}_i^*(\omega) \cdot (f_i^*(\omega) - e_i(\omega))$$

and appeal to Definition 2.1 (iii). Finally, notice that in case (b) the first part of the proposition, proven above, gives

$$\sum_{\omega\in F} \max_{1\le j\le d} R^*(\omega)_j \mathbb{P}(\omega) \le 0$$

for every $F \in \mathcal{F}_N$. Here $R^*(\omega) := \sum_{i\in I}(f_i^*(\omega) - e_i(\omega))$ is now a $\mathcal{F}_N$-measurable function, so we may apply the inequality in particular to the set F consisting of all $\omega \in \Omega$ with $\max_j (R^*(\omega))_j > 0$. This gives $\mathbb{P}(F) = 0$, so $F = \emptyset$ and the desired inequality follows.

Finally, observe that if $u_i(\omega, \cdot)$ is strongly monotone for every $i \in I$ and $\omega \in \Omega$, then so is $v_i(\omega, \cdot)$. Therefore Definition 2.1 (iii) implies (1) budget balancedness and (2) strict positivity of the price vector. Thus, because of $\hat{p}_i^*(\omega) = p^*(\omega)$ in case (a), we have for every $\omega \in \Omega$ (1') $p^*(\omega) \cdot (f_i^*(\omega) - e_i(\omega) = 0$ for every i and (2') $p^*(\omega) \in \mathbb{R}^d_{++}$. Together with the free disposal feasibility, already established, this implies the desired market clearing feasibility in case (a). As for case (b), a simple modification of the above proof of free disposal feasibility, exploiting (1), shows that (5.1) now holds with equality. Since (2') is now directly postulated, the rest of the feasibility proof is as just given.

5.2 An auxiliary result and its proof

To facilitate the proof of Theorem 4.1, this subsection is devoted to an auxiliary result and its proof.

Proposition 5.1 *Suppose that Assumptions 4.1–4.3 hold and that the set $X_i(\omega)$ is actually compact for every $i \in I$ and $\omega \in \Omega$. Then there exists a Bayesian–Walrasian equilibrium pair for the differential information economy $\mathcal{E}$.*

To prove this proposition, we consider the following multifunction Φ from $\prod_{i\in I} L_{X_i}(\mathcal{F}_i) \times \Delta^\Omega$ into itself: for $(f_i)_{i\in I} \in \prod_{i\in I} L_{X_i}(\mathcal{F}_i)$ and $p \in \Delta^\Omega$ we define $\Phi((f_i)_{i\in I}), p)$ to be the set of all $((g_i)_{i\in I}, q')$ such that

$$g_i(\omega) \in \operatorname*{argmax}_{x_i\in \hat{B}_i(\omega,p)} v_i(\omega, x_i) \quad \text{for every } i \in I \text{ and } \omega \in \Omega$$

and

$$q' \in \operatorname*{argmax}_{q\in\Delta^\Omega} \sum_{\omega\in\Omega} q(\omega) \cdot \sum_{i\in I} (f_i(\omega) - e_i(\omega))\mathbb{P}(\omega).$$

Observe that for every $i \in I$ the set $L_{X_i}(\mathcal{F}_i)$ contains e_i; hence it is nonempty. Observe also that the set $\prod_{i\in I} L_{X_i}(\mathcal{F}_i) \times \Delta^\Omega$ is clearly nonempty, convex and compact (compactness holds by Tychonov's theorem, in view of the extra hypothesis). Moreover, observe that each set $\Phi((f_i)_{i\in I}, p)$ is obviously nonempty (by Weierstrass' theorem), convex (by concavity) and closed (by continuity). It is a standard exercise to prove that the graph of Φ is also closed, given Assumptions 4.1 and 4.2. Because of the compactness of $\prod_{i\in I} L_{X_i}(\mathcal{F}_i) \times \Delta^\Omega$, this implies that the multifunction Φ is upper hemicontinuous. Hence, by Kakutani's fixed point theorem, there exist $(f_i^*)_{i\in I} \in \prod_{i\in I} L_{X_i}(\mathcal{F}_i)$ and $p^* \in \Delta^\Omega$ such that

$$f_i^*(\omega) \in \operatorname*{argmax}_{x_i \in \hat{B}_i(\omega, p^*)} v_i(\omega, x_i) \quad \text{for every } i \in I \text{ and } \omega \in \Omega, \tag{5.2}$$

and such that for $R^*(\omega) := \sum_{i\in I}(f_i^*(\omega) - e_i(\omega))$,

$$L := \max_{q\in\Delta^\Omega} \left[\sum_{\omega\in\Omega} q(\omega) \cdot R^*(\omega)\mathbb{P}(\omega)\right] = \sum_{\omega\in\Omega} p^*(\omega) \cdot R^*(\omega)\mathbb{P}(\omega) =: R.$$

As for L, it is easily seen that

$$L = \sum_{\omega\in\Omega} \left[\max_{\lambda\in\Delta} \lambda \cdot R^*(\omega)\right] \mathbb{P}(\omega) = \sum_{\omega\in\Omega} \max_{1\le j\le d} (R^*(\omega))_j \mathbb{P}(\omega).$$

So we obtain

$$0 = L - R = \sum_{\omega\in\Omega} \left[\max_{1\le j\le d} (R^*(\omega))_j - p^*(\omega) \cdot R^*(\omega)\right] \mathbb{P}(\omega).$$

Because all summands in the above expression are nonnegative and because $\mathbb{P}(\omega) > 0$ for every $\omega \in \Omega$, we conclude that $p^*(\omega) \cdot R^*(\omega) = \max_{1\le j\le d}(R^*(\omega))_j$ for every $\omega \in \Omega$. □

5.3 Proof of Theorem 4.1

Next, we use Proposition 5.1 to prove Theorem 4.1. This goes by a standard truncation argument. For $m \in \mathbb{N}$, $i \in I$ and $\omega \in \Omega$ we let $X_i^m(\omega)$ be the set of all $x \in X_i(\omega)$ such that

$$\sum_{j=1}^{d} (x)_j \le m \sum_{\omega\in\Omega} \sum_{j=1}^{d} (e_i(\omega))_j =: \gamma_i^m.$$

As a consequence of Assumption 4.1, this set is compact and convex. It is also nonempty, for it contains $e_i(\omega)$ and in fact, by Assumption 4.2 it contains an open neighborhood of $e_i(\omega)$ if $m \ge 2$ (because $\sum_j (y)_j < \sum_{j=1}^{d}(e_i(\omega))_j$ for $y \in \mathbb{R}^d_+$ implies that $\sum_{j=1}^{d}(e_i(\omega)+y)_j < \gamma_i^2$ – here $\sum_{j=1}^{d}(e_i(\omega))_j$ is strictly positive because of Assumption 4.2). Hence, as a consequence of Proposition 5.1, there exists for every $m \geq 2$ a pair (p^m, f^m) in $\Delta^\Omega \times \Pi_{i\in I} L_{X_i^m}(\mathcal{F}_i)$ such that

$$f_i^m(\omega) \in \operatorname*{argmax}_{x_i \in \hat{B}_i(\omega, p^m)} v_i(\omega, x_i) \quad \text{for every } \omega \in \Omega \text{ and every } i \in I \tag{5.3}$$

and

$$p^m(\omega) \cdot \sum_{i\in I}(f_i(\omega) - e_i(\omega)) = \max_{1\leq j\leq d} \sum_{i\in I}(f_i^m(\omega) - e_i(\omega))_j] \quad \text{for every } \omega \in \Omega. \tag{5.4}$$

Rather than extracting suitable subsequences, we can suppose without loss of generality that the sequences $\{p^m\}_m \subset \Delta^\Omega$ and $\{f^m\}_m$, thus obtained, are pointwise convergent, i.e.,

$$p^*(\omega) := \lim_{m\to\infty} p^m(\omega) \in \Delta \text{ exists} \quad \text{for every } \omega \in \Omega$$

and for every $i \in I$,

$$f_i^*(\omega) := \lim_{m\to\infty} f_i^m(\omega) \in X_i(\omega) \text{ exists} \quad \text{for every } \omega \in \Omega.$$

This is because Δ^Ω is obviously compact and because (5.4) implies

$$\sum_{\omega\in\Omega}\left[\max_{1\leq j\leq d}\sum_{i\in I}(f_i^m(\omega))_j\right]\mathbb{P}(\omega) \leq \sum_{\omega\in\Omega}\sum_{j=1}^{d}\sum_{i\in I} e_i(\omega))_j < +\infty.$$

by the first part of Proposition 2.1. This causes the sequence $\{f_i^m(\omega)\}_m$ to be bounded in $\mathbb{R}_+^d$ for every $i \in I$ and $\omega \in \Omega$. By a standard limit argument we get from (5.3) in a first stage that for every $i \in I$ the inequality

$$v_i(\omega, f_i^*(\omega)) \geq v_i(\omega, x_i) \tag{5.5}$$

holds for every $\omega \in \Omega$ and for every $x_i \in X_i(\omega)$ with $\hat{p}_i^*(\omega) \cdot x_i < \hat{p}_i^* \cdot e_i(\omega)$ (observe from (2.2) that $\hat{p}_i^m(\omega) \to \hat{p}_i^*(\omega)$). In a second stage, such validity of (5.5) is extended to all $\omega \in \Omega$ with and $x_i \in \hat{B}_i(\omega, p^*)$, by forming convex combinations $y_i(\theta) := (1-\theta)x_i + \theta e_i(\omega)$ with $\theta \downarrow 0$ (observe that Assumption 4.2 guarantees $\hat{p}_i^*(\omega) \cdot y_i(\theta) < \hat{p}_i^*(\omega) \cdot e_i(\omega)$). Thus, Definition 2.1 (iii) obtains. Finally, in the limit (5.4) immediately leads to Definition 2.1 (iv).

6 Concluding remarks

We introduced a new equilibrium notion, called here Bayesian Walrasian equilibrium. Our preliminary research shows that in simple standard situations our new concept seems to be a good alternative to the REE, as it is not susceptible to the existence problems of the REE and passes the Kreps criticism (Kreps 1977). However, we believe that more research is needed in this direction.

Let us say at the outset that what makes interim expected utility decision quite difficult to analyze, is the fact that, in the interim stage, risk sharing is limited. In particular, the resulting equilibrium allocations must be interim individually rational and once private measurability assumptions enter the utility functions, allocations must be ex post individually rational. Thus, the possibilities for risk sharing are not there, so to come up with new interim equilibrium notions is not an easy task. To reinforce this point, consider the REE, but allow agents not to condition their interim utility functions on the information that the equilibrium prices generate, but only on their own private information.[5] The Kreps example still shows that such equilibrium need not exist. However, more is true. No private information interim individually rational and interim Pareto optimal allocation is expected to exist in general (Hahn and Yannelis 1997, Sect. 8.2).

The above discussion suggests, that to insist on predicting market clearing (exact feasibility of allocations) in an interim stage, is perhaps too much to expect. We think that it would be possible to have exact market clearing (feasibility), only if the economy is repeated from period to period and agents refine their information after they observe the BWE allocations. As the time goes to infinity, it is possible that agents have learned everything they needed to know, i.e., their partitions may become identical (e.g., composed of all singletons), and in this case the BWE is certainly an ex post WE. We plan to work on this conjecture in a future paper. Two other important issues that need to be addressed are: first, the incentive compatibility of the BWE and, second, the implementation of the BWE as a perfect Bayesian equilibrium of an extensive form game. The latter will provide the dynamics of the BWE and will make transparent how BWE allocations are reached. At the moment all these issues are open questions.

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[5] I.e., prices reveal no information at all. This may be the case in a non-revealing REE (see Glycopantis and Yannelis 2007).

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