LECTURE NOTES IN GENERAL EQUILIBRIUM THEORY [1]

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Part I

MATHEMATICS

1 Topological Space

DEFINITION 1.1: A **topology** τ on a set X is a collection of subsets of X satisfying:

(1) $\emptyset, X \in \tau$,

(2) τ is closed under finite intersections.

(3) τ is closed under arbitrary unions.

The pair (X, τ) is called a **topological space**. We call a member of τ an **open set** in X. The complement of an open set is a **closed set**.

DEFINITION 1.2: Let (X, τ) be a topological space, and let A be any subset of X.

(1) The **interior** of A, denoted by $intA$, is the largest open set included in A.

(2) The **closure** of A, denoted by $\bar{A}$, is the smallest closed set including A.

(3) A **neighborhood** of a point x is any set V containing x in its interior. In this case we say that x is an **interior point** of V.

(4) A point x is **closure point** of the set A if every neighborhood of x meets A. Note that $\bar{A}$ coincides with the set of all closure points of A.

(5) A point x is **accumulation point** (or a **limit point**, or a **cluster point**) of the set A if every neighborhood V of x we have $(V \setminus \{x\}) \cap A \neq \emptyset$. The set of all accumulation points of A is denoted by A'.

(6) A point x is a **boundary point** of A if each neighborhood V of x satisfies both $V \cap A \neq \emptyset$ and $V \cap A^c \neq \emptyset$. The set of all boundary points of A is denoted by ∂A.

DEFINITION 1.3: A function $f : X \to Y$ between two topological spaces is **continuous** if $f^{-1}(U)$ is open for every open set U. We say that f is **continuous at the point** x if $f^{-1}(V)$ is a neighborhood of x whenever V is a neighborhood of $f(x)$.

NOTE: In a metric space, continuity at a point x reduces to the familiar $\varepsilon - \delta$ definition: For every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$.

THEOREM 1.1: For a function $f : X \to Y$ between topological spaces, the following are equivalent :

(1) f is continuous on X.

(2) If C is an closed subset of Y, then $f^{-1}(C)$ is an closed subset of X.

(3) For every subset B of X, $f^{-1}(intB) \subset int[f^{-1}(B)]$.

(4) For every subset A of X, $f(\bar{A}) \subset \overline{f(A)}$.

DEFINITION 1.4: An **open cover** of a set K is a collection of open sets whose union includes K. A subset K of a topological space is **compact** if every open cover of K includes a finite subcover. That is, K is compact if every family $\{V_i : i \in I\}$ of open sets satisfying $K \subset \bigcup_{i \in I} V_i$ has a finite subfamily $V_{i_1}, V_{i_2}, \cdots, V_{i_n}$ such that $K \subset \bigcup_{j=1}^{n} V_{i_j}$.

THEOREM 1.2: Every continuous function between topological spaces carries compact sets to compact sets.

Proof: Let $f : X \to Y$ be a continuous function between two topological spaces, and let K be a compact subset of X. Also, let $\{V_i : i \in I\}$ be an open cover of $f(K)$. Then $\{f^{-1}(V_i) : i \in I\}$ is an open cover of K. By the compactness of K, there exists $i_1, \ldots, i_n$ satisfying $K \subset \bigcup_{j=1}^{n} f^{-1}(V_{i_j})$. Hence,

$$f(K) \subset f(\bigcup_{j=1}^{n} f^{-1}(V_{i_j})) = \bigcup_{j=1}^{n} f(f^{-1}(V_{i_j})) \subset \bigcup_{j=1}^{n} V_{i_j}$$

which shows that $f(K)$is a compact subset of Y.

Corollary (Weierstrass) : A continuous real-valued function defined on a compact space achieves its maximum and minimum values.

2 Metric Space

DEFINITION 2.1: A **metric** on a set X is a function $d : X \times X \to \Re$ satisfying:

(1) $d(x, y) \geq 0$.

(2) $d(x, y) = 0$ iff $x = y$.

(3) $d(x, y) = d(y, x)$.

(4) $d(x, y) + d(y, z) \geq d(x, z)$.

The pair (X, d) is called a **metric space**.

Given a metric d, let $B_\varepsilon(x) = \{y : d(x, y) < \varepsilon\}$, the **open ε-ball** around x. A set U is open in the **metric topology** generated by d if for each point x in U there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$. A topological space is **metrizable** if there exists a metric d on X generating the topology of X.

The **Euclidean metric** on $\Re^n$, $d(x, y) = \{\sum_{i=1}^n (x_i - y_i)^2\}^{1/2}$, defines its usual topology, called **Euclidean topology**.

The metric, a real-valued function, allows us to analyze spaces using what we know about the real numbers. The distinguishing features of the theory of the metric spaces, which are absent from the theory of topology, are the notions of *uniform continuity* and *completeness*.

DEFINITION 2.2: For a nonempty subset A of a metric space (X, d), its **diameter** is defined by diam $A = sup\{d(x, y) : x, y \in A\}$. A set A is **bounded** if diam $A < \infty$.

THEOREM 2.1(Heine-Borel) : Subsets of $\Re^n$ are compact if and only if they are closed and bounded.

From now on, let the space be R^n (Euclidean space).

DEFINITION 2.3: Define $R^\ell = \{(x_1, x_2, \ldots, x_\ell) : x_i \in R, i = 1, 2, \ldots, \ell\}$ and let $x \in R^\ell$ and $y \in R^\ell$.

(1) $x \leq y$ means $x_i \leq y_i$ for every $i = 1, \ldots, \ell$.

(2) $x < y$ means $x_i \leq y_i$ and $x \neq y$.

(3) $x \ll y$ means $x_i < y_i$ for every $i = 1, \ldots, \ell$.

(4) $R^\ell_+ = \{x \in R^\ell : x \geq 0\}$.

(5) $R^\ell_{++} = \{x \in R^\ell : x \gg 0\}$.

(6) $R^\ell_- = \{x \in R^\ell : x \leq 0\}$.

(7) The **sum** of sets $X_1 \subset R^\ell$, $X_2 \subset R^\ell$ is defined by $X_1 + X_2 = \{x_1 + x_2 : x_i \in X_i, i = 1, 2\}$.

(8) Let $\alpha \in R$ and $X \subset R^\ell$. $\alpha X = \{\alpha x : x \in X\}$.

(9) The **product** of sets $X_1 \subset R^\ell$, $X_2 \subset R^\ell$ is defined by $\prod_{i=1}^2 X_i = X_1 \times X_2 = \{(x_1, x_2) : x_i \in X_i, i = 1, 2\}$.

(10) The **dot product** of x and y is definded by $x \cdot y = \sum_{k=1}^\ell x_k y_k$.

(11) The **Euclidean norm** $\|x\|$ of x is defined by $\|x\|^2 = x \cdot x$.

THEOREM 2.2: Let X and Y be sets and let $f : X \to Y$ be a function. Let A and A_i's be subsets of X, and B and B_i's be subsets of Y. Then the following hold:

(1) $\bigcap_{i\in I} f^{-1}(B_i) = f^{-1}(\bigcap_{i\in I} B_i)$.

(2) $\bigcup_{i\in I} f^{-1}(B_i) = f^{-1}(\bigcup_{i\in I} B_i)$.

(3) $X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$.

(4) $\bigcup_{i\in I} f(A_i) = f(\bigcup_{i\in I} A_i)$.

(5) $f(\bigcap_{i\in I} A_i) \subset \bigcap_{i\in I} f(A_i)$.

(6) $f(f^{-1}(B)) \subset B$, and $f(f^{-1}(B)) = B$ iff f is onto.

(7) $A \subset f^{-1}(f(A))$, and $A = f^{-1}(f(A))$ iff f is one-to-one.

DEFINITION 2.4: If $x \in R^\ell$, then the **open ball** at x with radius $\varepsilon > 0$ is the set

$$B_\varepsilon(x) = \{x' \in R^\ell : d(x, x') < \varepsilon\}.$$

DEFINITION 2.5: A subset S of R^ℓ is **open** if for every $x \in S$, there exists a open ball $B_\varepsilon(x) \subset S$.

DEFINITION 2.6: A subset S of R^ℓ is **open relative to (in)** X if there exists an open subset A of R^ℓ such that $S = A \cap X$.

DEFINITION 2.7: A point $x \in R^\ell$ is an **interior point** of $S \subset R^\ell$ if there exists an open ball $B_\varepsilon(x) \subset S$. The set of all interior points of S is the **interior** of S and is denoted by $intS$.

DEFINITION 2.8: A **neighborhood** U of $x \in R^\ell$ is a subset which contains an open set B containing x. A **neighborhood** U of $S \subset R^\ell$ is a subset which contains an open set B containing S.

DEFINITION 2.9: A subset S of R^ℓ is **closed** if its complement is an open set.

DEFINITION 2.10: A subset S of R^ℓ is **closed relative to (in)** X if there exists a closed subset A of R^ℓ such that $S = A \cap X$.

DEFINITION 2.11: A point $x \in R^\ell$ is a **closure point (adherent point)** of $S \subset R^\ell$ if every open ball at x contains at least one element of S. The set of all closure points of S is the closure of S and is denoted by $\bar{S}$.

DEFINITION 2.12: A point $x \in R^\ell$ is an **accumulation point (cluster point, limit point)** of $S \subset R^\ell$ if every open ball at x contains one element of S which is distinct from x. The set of all accumulation points of S is the **derived set** of S and is denoted by S'.

N. B. Note that x need not be an element of S. Clearly, every accumulation point of a set must be a closure point of that set. It should be clear that $\bar{S} = S \cup S'$. In particular, it follows that a set is closed iff it contains its accumulation points.

DEFINITION 2.13: A sequence $\{x_n\}$ in R^ℓ is **convergent** to x in R^ℓ if

$$\lim_{n\to\infty} d(x_n, x) = 0$$

We write $\lim_n x_n = x$ or $x_n \to x$.

THEOREM 2.3: Let S be a subset of R^ℓ. Then a point $x \in R^\ell$ belongs to $\bar{S}$ iff there exists a sequence $\{x_n\}$ of S such that $x_n \to x$. In particular, if x is an accumulation point of S, then there exists a sequence of S with distinct terms that converges to x.

N. B. A subset S of R^ℓ is closed iff the limit of every convergent sequence in S belongs to S.

DEFINITION 2.14: A point x is a **boundary point** of $S \subset R^\ell$ if every open ball of x has a nonempty intersection with S and $R^\ell \setminus S$. The set of all boundary points of S

is the **boundary** of S and is denoted by ∂S. **N. B.** By the symmetry of the definition, $\partial S = \partial(S^c)$. Also, a simple argument shows that $\partial S = \bar{S} \cap \bar{S}^c$.

DEFINITION 2.15: A subset S of R^ℓ is **bounded** if there are two points x' and x'' in R^ℓ such that $x' \leq x \leq x''$ for every $x \in S$.

THEOREM 2.4: A bounded sequence has a convergent subsequence.

DEFINITION 2.16: Let $X \subset R^\ell$ and $Y \subset R^m$. A function $f : X \to Y$ is **continuous at** $x \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(x')) < \varepsilon$ whenever $d(x, x') < \delta$. A function f is **continuous** on X if f is continuous at every point of X.

DEFINITION 2.17: A family of subsets $\{A_i : i \in I\}$ of R^ℓ is a **cover** of $S \subset R^\ell$ if $S \subset \bigcup_{i\in I} A_i$. If a subfamily of $\{A_i : i \in I\}$ also covers S, then it is a **subcover**. Any cover of S consisting of open sets is an **open cover** of S.

DEFINITION 2.18: A subset S of R^ℓ is **compact** if every open cover of S can be reduced to a finite subcover.

THEOREM 2.5: Let S be a subset of R^ℓ. The following are equivalent :

(1) S is compact.

(2) S is closed and bounded.

(3) Every sequence of S has a convergent subsequence whose limit belongs to S.

(4) Every infinite subset of S has an accumulation point in S.

(5) Every collection of closed subsets of S with the finite intersection property (*i.e.*, every finite subcollection has a nonempty intersection) has a nonempty intersection.

THEOREM 2.6:

(1) Every closed subset of a compact set is compact.

(2) If $f : X \to Y$ is continuous, and K is compact in X, then $f(K)$ is compact in Y.

(3) S_i is compact for every $i \in I$ iff $\prod_{i\in I} S_i$ is compact.

(4) S_i is compact for every $i = 1, \ldots, m$ iff $\sum_{i=1}^m S_i$ is compact.

3 Convex Sets

DEFINITION 3.1: A subset S of R^ℓ is **convex** if for $x, x' \in S$, $x^\alpha = \alpha x + (1-\alpha)x' \in S$ for every $\alpha \in [0,1]$.

DEFINITION 3.2: x is an **extreme point** if $x = \alpha x' + (1-\alpha)x''$ with $\alpha \in (0,1)$ implies $x = x' = x''$.

DEFINITION 3.3: $\sum_{i=1}^m \alpha x_i$ is a **(finite) convex combination** of $x_1, x_2, \ldots, x_m$ if $\alpha_1, \alpha_2, \ldots, \alpha_m$ satisfies $\sum_{i=1}^m \alpha_i = 1$ and $\alpha_i \geq 0$ for every i. A **strict positive convex combination** is a convex combination where $\alpha_i > 0$ for every i.

DEFINITION 3.4: The **convex hull** of $S \subset R^\ell$ is the set of all finite convex combinations from S and is denoted by coS.

THEOREM 3.1 (Carathéodory): Let S be a subset of R^ℓ. Then, every point $x \in coS$ is a convex combination of $\ell + 1$ points in S.

THEOREM 3.2 (Krein-Milman): Let S be a nonempty compact convex subset of R^ℓ. Then $S = co(exS)$ where exS is the set of extreme points.

THEOREM 3.3 (Shapley-Folkman): Let S_i be nonempty subsets of R^ℓ for every $i = 1, 2, \ldots, m$. For every $x \in co(\sum_{i=1}^m S_i)$, there exist $x_i \in coS_i$, $i = 1, 2, \ldots, m$ such that $x = \sum_i x_i$ and $\#\{i : x_i \notin S_i\} \leq \ell$.

THEOREM 3.4:

(1) If S_i is convex for every $i \in I$, so is $\bigcap_{i \in I} S_i$.

(2) If S_i is convex for each $i = 1, \ldots, m$, so are $\sum_i S_i$ and $\prod_i S_i$.

(3) Let $\alpha \in R$. If S is convex, so is αS.

(4) If S is convex, so are $intS$ and $\bar{S}$.

(5) If S is open (compact), so is coS.

(6) $coS := \bigcap \{C \subset R^\ell : C \text{ is convex and } S \subset C\}$.

(7) $co\bar{S} \subset \overline{coS}$

(8) $co(\sum_i S_i) = \sum_i coS_i$.

N. B. It follows from (6) that coS is the smallest convex set containing S.

EXAMPLE 3.1: Consider

$$S = \{(x,y) \in R^2 : y \geq \frac{1}{|x|}\}.$$

Then S is closed, but S is not compact and coS is not closed.

DEFINITION 3.5: A **hyperplane** in R^ℓ is a set $\{x \in R^\ell : p \cdot x = \alpha\}$ where $p \in R^\ell \setminus \{0\}$ and $\alpha \in R$. We denote it by $H(p,\alpha)$. The vector p is **normal** to the hyperplane $H(p,\alpha)$.

N. B. A hyperplane is the set of solutions of one linear equation in ℓ variables.

DEFINITION 3.6: A set $\{x : p \cdot x \leq \alpha\}$ is a **closed lower half space** $H(p,\alpha)$. A set $\{x : p \cdot x < \alpha\}$ is an **open lower half space** $H(p,\alpha)$.

DEFINITION 3.7: Two sets A and B in R^ℓ are **separated** by a hyperplane $H(p,\alpha)$ if there are $p \in R^\ell \setminus \{0\}$ and $\alpha \in R$ such that for every $x \in A$ and $y \in B$

$$p \cdot x \leq \alpha \leq p \cdot y$$

They are **strictly separated** if the inequalities are replaced by strict inequalities. They are **strongly separated** if $\sup_{x \in A} p \cdot x < \alpha < \inf_{y \in B} p \cdot y$.

THEOREM 3.6 (Separating Hyperplane Theorem):

(1) Let S be a nonempty closed and convex subset in R^ℓ and $z \notin S$. Then there exists a point $x^* \in S$ and a hyperplane $H(p,\alpha)$ through x^* such that

$$p \cdot z < \alpha = p \cdot x^* = \inf_{x \in S} p \cdot x.$$

(2) Let S be a nonempty convex set of R^ℓ and $z \notin S$. Then there exists a hyperplane $H(p,\alpha)$ through z such that for every $x \in S$

$$p \cdot z = \alpha \leq p \cdot x$$

(3) Let A and B be disjoint nonempty convex subsets of R^ℓ. Let A be closed and B be compact. Then A and B can be strongly separated by a hyperplane.

(4) Let A and B be disjoint nonempty convex subsets of R^ℓ. Let A and B be closed. Then A and B can be strictly separated by a hyperplane.

(5) Let A and B be disjoint nonempty convex subsets in R^ℓ. Then there exists a hyperplane separating the sets A and B.

N. B. In (2), if z is on the boundary of S, the hyperplane is called **supporting hyperplane**.

DEFINITION 3.8: Let X be a convex subset of R^ℓ and let $f : X \to R$ be a function.

(1) A function f is **concave** if for $x, x' \in X$, $f(\alpha x + (1-\alpha)x') \geq \alpha f(x) + (1-\alpha)f(x')$ for $\alpha \in [0, 1]$.

(2) A function f is **convex** if $-f$ is concave.

(3) A function f is **quasi-concave** if $\{x \in X : f(x) \geq \alpha\}$ is convex for every $\alpha \in R$.

(4) A function f is **quasi-convex** if $(-f)$ is quasi-concave.

4 Correspondences

A correspondence is a set-valued function and arise naturally in many economic applications, for instance, budget correspondence, excess demand correspondence, etc. The biggest difference between functions and correspondences has to do with the definitions of an inverse image. The inverse image of a set A under a function f is the set $\{x : f(x) \in A\}$. For a correspondence, there are two reasonable generalizations, the upper inverse and the lower inverse. Having two definitions of the inverse leads to two definitions of continuity, that is, lower hemicontinuity and upper hemicontinuity.

Let $X \subset R^\ell$ and $Y \subset R^m$.

DEFINITION 4.1: A **correspondence** $\varphi : X \to 2^Y$ is a function from X to the family of all subsets of Y.

A correspondence $\varphi : X \to 2^Y$ is compact-valued (nonempty-valued, convex-valued, open-valued, closed-valued, bounded-valued) if $\varphi(x)$ is a compact (nonempty, convex, open, closed, bounded) subset of Y for every $x \in X$.

DEFINITION 4.2: The **graph** of a correspondence $\varphi : X \to 2^Y$ is defined by

$$G_\varphi = \{(x, y) \in X \times Y : y \in \varphi(x)\}.$$

A correspondence $\varphi : X \to 2^Y$ **has open (closed) graph** if the set $G := \{(x, y) \in X \times Y : y \in \varphi(x)\}$ is open (closed) in $X \times Y$.

DEFINITION 4.3: Let $\varphi : X \to 2^Y$ be a correspondence, $A \subset X$, and $B \subset Y$.

(1) The **image** of A by φ is defined by $\varphi(A) = \bigcup_{x \in A} \varphi(x)$.

(2) The **inverse** of B by φ is defined by $\varphi^{-1}(B) = \{x \in X : \varphi(x) = B\}$.

(3) The **upper inverse** of B by φ is defined by $\varphi^{+}(B) = \{x \in X : \varphi(x) \subset B\}$.

(4) The **lower inverse** of B by φ is defined by $\varphi^{-}(B) = \{x \in X : \varphi(x) \cap B \neq \emptyset\}$.

(5) The **upper section** of φ at x is defined by $\varphi(x)$.

(6) The **lower section** of φ at y is defined by $\varphi^{-1}(y) = \{x \in X : y \in \varphi(x)\}$.

NOTE : $\varphi^{-1}(y) = \varphi^{-}(\{y\})$.

THEOREM 4.1: Let $\varphi : X \to 2^Y$ be a correspondence and $B \subset Y$.

(1) $\varphi^{-1}(B) \subset \varphi^{+}(B) \subset \varphi^{-}(B)$.

(2) $\varphi^{+}(B^c) = [\varphi^{-}(B)]^c$.

(3) $\varphi^{-}(B) = \bigcup_{y \in B} \varphi^{-}(\{y\}) = \bigcup_{y \in B} \varphi^{-1}(y)$.

DEFINITION 4.4: Let $\varphi : X \to 2^Y$ be a correspondence.

(1) φ **has open (closed) upper sections** if $\varphi(x)$ is open (closed) for every $x \in X$.

(2) φ **has open (closed) lower sections** if $\varphi^{-1}(y)$ is open (closed) for every $y \in Y$.

(3) φ **has open (closed) sections** if it has both open (closed) upper sections and open (closed) lower sections.

N. B. φ has open (closed) upper sections iff φ is open-valued (closed-valued).

EXAMPLE 4.1: Define the correspondence $P : X \to 2^X$ by $P(x) := \{x' \in X : x' \succ x\}$ and $P^{-1} : X \to 2^X$ by $P^{-1}(x) := \{x' \in X : x \succ x'\}$. The upper section $P(x)$ of P at

x is the upper contour set of $\succ$ at x and the lower section $P^{-1}(x)$ of P at x is the lower contour set of $\succ$ at x.

DEFINITION 4.5: Let $\varphi : X \to 2^Y$ be a correspondence.

(1) φ is **closed at** x if $(x_n, y_n) \to (x, y)$ and $y_n \in \varphi(x_n)$ for every n imply $y \in \varphi(x)$. It is **closed** (**has closed graph**) if it is closed at every $x \in X$.

(2) φ is **upper hemi-continuous (u.h.c.) at** x if, for every open set V containing $\varphi(x)$, there exists a neighborhood U of x such that $\varphi(x') \subset V$ for every $x' \in U$. φ is **upper hemi-continuous** if it is upper hemi-continuous at every $x \in X$.

(3) φ is **lower hemi-continuous (l.h.c.) at** x if, for every open set V with $\varphi(x) \cap V \neq \emptyset$, there exists a neighborhood U of x such that $\varphi(x') \cap V \neq \emptyset$ for every $x' \in U$. A correspondence φ is **lower hemi-continuous** if it is lower hemi-continuous at every $x \in X$.

(4) φ is **continuous at** x if φ is both upper hemi-continuous and lower hemi-continuous at x. It is **continuous** if it is continuous at every $x \in X$.

THEOREM 4.2: Let $\varphi : X \to 2^Y$ be a correspondence. The following are equivalent.

(1) φ is upper hemi-continuous.

(2) For each open subset B of Y, $\varphi^+(B)$ is open.

(3) For each closed subset C of Y, $\varphi^-(C)$ is closed.

THEOREM 4.3: Let $\varphi : X \to 2^Y$ be a correspondence. The following are equivalent.

(1) φ is lower hemi-continuous.

(2) For each open subset B of Y, $\varphi^-(B)$ is open.

(3) For each closed subset C of Y, $\varphi^+(C)$ is closed.

COROLLARY : Let $\varphi : X \to 2^Y$ be a correspondence.

(1) If φ is upper hemi-continuous, then $\{x \in X : \varphi(x) \neq \emptyset\}$ is closed.

(2) If φ is lower hemi-continuous, then $\{x \in X : \varphi(x) \neq \emptyset\}$ is open.

THEOREM 4.4: Let $\varphi : X \to 2^Y$ be a correspondence.

(1) Let φ be compact-valued and upper hemi-continuous. If K is compact, then $\varphi(K)$ is compact (closed).

(2) If φ has open (closed) graph, then it has open (closed) sections.

(3) If φ has open lower sections, then it is lower hemi-continuous.

(4) If φ has open graph, then it is lower hemi-continuous.

(5) If φ is singleton-valued at x and either upper hemi-continuous or lower hemi-continuous at x, then it is continuous at x.

N. B. Note that (4) is a corollary of (2) and (3).

THEOREM 4.5: An upper hemicontinous correspondence $\varphi : X \to 2^Y$ is closed if either:

(1) φ is closed-valued and Y is regular [1], or

(2) φ is compact-valued and Y is Hausdorff [2].

For a correspondence having a compact Hausdorff range, the properties of being closed and being upper hemicontinous coincide.

THEOREM 4.6(Closed Graph Theorem): A closed-valued correspondence with compact Hausdorff range is closed if and only if it it upper hemicontinuous.

Similarly, φ is lower hemi-continuous at x iff $x_n \to x$ and $y \in \varphi(x)$ imply that there exists a sequence $\{y_n\}$ such that $y_n \in \varphi(x_n)$ for every n and $y_n \to y$.

EXAMPLE 4.2: Consider a correspondence $\varphi : R \to 2^R$.

(1) Define φ by

$$\varphi(x) = \begin{cases} 1/x & \text{if } x > 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

Then φ has closed graph and compact-valued, but is not upper hemi-continuous.

[1] A topological space is **regular** if every nonempty closed set and every singleton disjoint from it can be separated by open sets

[2] A topology is called **Hausdorff** if any two distinct points can be separated by disjoint neighborhood of the points. That is, for each pair $x, y \in X$ with $x \neq y$ there exist neighborhoods $U \in N_x$ and $V \in N_y$ such that $U \cap V = \emptyset$

(2) Let $\varphi(x) = (0, 1)$. Then it is upper hemi-continuous but does not have closed graph.

THEOREM 4.7 (Closure): Let $\varphi : X \to 2^Y$ be a correspondence. Define the correspondence $\bar{\varphi} : X \to 2^Y$ by $\bar{\varphi}(x) = \bar{\varphi(x)}$.

(1) If φ is upper hemi-continuous at x, so is $\bar{\varphi}$.

(2) φ is lower hemi-continuous at x iff $\bar{\varphi}$ is lower hemi-continuous at x.

EXAMPLE 4.3: Consider $\varphi : R \to 2^R$ with $\varphi(x) = \{x\}^c$. It is not upper hemi-continuous but its closure is upper hemi-continuous.

THEOREM 4.8 (Intersection): Let φ, μ, and φ_i's be correspondences from X to Y. Define the correspondence $\bigcap_{i\in I} \varphi_i : X \to 2^Y$ by $(\bigcap_{i\in I} \varphi_i)(x) = \bigcap_{i\in I} \varphi_i(x)$. Suppose the intersection is nonempty-valued.

(1) $G_{\cap\varphi_i} = \bigcap_{i\in I} G_{\varphi_i}$.

(2) If φ_i is closed-valued and upper hemi-continuous at x for every $i \in I$, so is $\bigcap_i \varphi_i$.

(3) If φ is lower hemi-continuous at x and μ has open graph, then $\varphi \cap \mu$ is lower hemi-continuous at x.

(4) If φ and μ have open (closed) sections, so does $\varphi \cap \mu$.

THEOREM 4.9 (Union): Let $\varphi_i : X \to 2^Y$ be a correspondence for every $i \in I$. Define the correspondence $\bigcup_{i\in I} \varphi_i : X \to 2^Y$ by $(\bigcup_{i\in I} \varphi_i)(x) = \bigcup_{i\in I} \varphi_i(x)$.

(1) $G_{\cup\varphi_i} = \bigcup_{i\in I} G_{\varphi_i}$.

(2) If φ_i is upper hemi-continuous (closed) at x for every $i = 1, \ldots, n$, so is $\bigcup_i \varphi_i$.

(3) If φ_i is lower hemi-continuous at x for every $i \in I$, so is $\bigcup_i \varphi_i$.

THEOREM 4.10 (Composition): Let $\varphi : X \to 2^Y$ and $\mu : Y \to 2^Z$ be correspondences. Define the correspendence $\mu \circ \varphi : X \to Z$ by $(\mu \circ \varphi)(x) = \bigcup_{y\in\varphi(x)} \mu(y)$.

(1) If φ and μ are upper semi-continuous at x, so is $\mu \circ \varphi$.

(2) If φ and μ are lower semi-continuous at x, so is $\mu \circ \varphi$.

THEOREM 4.11 (Product): Let $\varphi_i : X \to 2^{Y_i}$ be a correspondence for every $i \in I$. Define the correspondence $\prod_{i\in I} \varphi_i : X \to 2^Y$ by $(\prod_{i\in I} \varphi_i)(x) = \prod_{i\in I} \varphi_i(x)$, where $Y = \prod_{i\in I} Y_i$.

(1) If φ_i has open (closed) graph for every $i \in I$, so does $\prod_i \varphi_i$.

(2) If φ_i is compact-valued and upper hemi-continuous at x for every $i \in I$, so is $\prod_i \varphi_i$.

(3) If φ_i is lower hemi-continuous at x for $i = 1, \ldots, n$, so is $\prod_i \varphi_i$.

THEOREM 4.12 (Sum): Let $Y_i \subset R^\ell$ and $\varphi_i : X \to 2^{Y_i}$ be a correspondence for $i = 1, 2, \ldots, n$. Define the correspondence $\sum_{i=1}^m \varphi_i : X \to 2^Y$ by $(\sum_{i=1}^m \varphi_i)(x) = \sum_{i=1}^m \varphi_i(x)$, where $Y = \sum_i Y_i$.

(1) If φ_i is compact-valued and upper hemi-continuous at x for $i = 1, \cdots, n$, so is $\sum_i \varphi_i$.

(2) If φ_i is lower hemi-continuous at x for $i = 1, \ldots, n$, so is $\sum_i \varphi_i$.

(3) If φ_i has open (closed) graph for $i = 1, \ldots, n$, so does $\sum_i \varphi_i$.

THEOREM 4.13 (Convex Hull): Let Y be convex. Let $\varphi : X \to 2^Y$ be a correspondence. Define the correspondence $co\varphi : X \to 2^Y$ by $(co\varphi)(x) = co[\varphi(x)]$.

(1) If φ is compact-valued and upper hemi-continuous at x, so is $co\varphi$.

(2) If φ is lower hemi-continuous (has open graph, has open lower sections) at x, so is (does) $co\varphi$.

EXAMPLE 4.4: Consider a correspondence $\varphi : R \to 2^R$.

$$\varphi(x) = \begin{cases} \{0, 1/x\} & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

Here, φ has a closed graph but $co\varphi$ does not have a closed graph.

5 Maximum Theorem

THEOREM 5.1(Berge): Let $\varphi : X \to 2^Y$ be a continuous correspondence with nonempty, compact-valued, and suppose $f : Gr\varphi \to \Re$ is continuous. Then

(1) The "value function" $m : X \to \Re$ defined by $m(x) = \sup\{f(x,y) : y \in \varphi(x)\}$ is continuous and

(2) The correspondence $\mu : X \to 2^Y$ defined by

$$\mu(x) = \{y \in \varphi(x) : f(x,y) = m(x)\}$$

is upper hemicontinuous with nonempty, compact-valued.

6 KKM Theorem, Existence of Maximal Element

THEOREM 6.1(Knaster-Kuratowski-Mazurkewicz) : Let X be an arbitrary convex subset of R^l. For $x \in X$, let $F(x)$ be a closed set in R^l satisfying the following assumptions:

(1) For any arbitrary set of points $\{x_1, \ldots, x_n\}$ of X,

$$co\{x_i, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i).$$

(2) $F(x)$ is compact for at least one $x \in X$.

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

THEOREM 6.2(Existence of Maximal Element) : Let X be a nonempty compact convex subset of R^ℓ and a correspondence $P : X \to 2^X$ be a preference correspondence such that:

(1) $x \notin P(x)$ for all $x \in X$

(2) $P(x)$ is convex for all $x \in X$

(3) P has open lower sections

Then there exists $x' \in X$ such that $P(x') = \emptyset$.

NOTE (1),(2) can be replaced by $x \notin conP(x)$ for all $x \in X$.

REMARK
KKM Theorem $\Leftrightarrow$ Browder Fixed Point Theorem $\Leftrightarrow$ Existence of Maximal Elements Theorem

7 Selection Theorems

DEFINITION 7.1: A **selection** from $\varphi : X \to 2^Y$ is a function $f : X \to Y$ such that, for every $x \in X$, $f(x) \in \varphi(x)$. If X and Y are topological spaces, then we say that f is a **continuous selection** if f is a selection and is continuous.

THEOREM 7.2(Yannelis-Prabhakar): Let X be paracompact[3], Y be topological vector space a correspondence $\varphi : X \to 2^Y$ be nonempty-valued and convex-valued. If φ has open lower sections, then there exists a continuous selection of φ.

THEOREM 7.3(Michael): Let X be paracompact, Y be separable Banach space [4] and a correspondence $\varphi : X \to 2^Y$ be nonempty-valued and convex-valued. If φ is lower hemi-continuous, then there exists a continuous selection of φ.

N. B.

(a) If $Y = R^n$ then Theorem 5.1 is a corollary of Theorem 5.2.

(b) If Y is any arbitrary linear topological space then Theorem 5.1 does not follow from Theorem 5.2.

8 Fixed Point Theorems

DEFINITION 8.1:

(1) Let $f : X \to X$ be a function. A **fixed point** of f is a point $x^* \in X$ such that $x^* = f(x^*)$.

(2) Let $\varphi : X \to 2^X$ be a correspendence. A **fixed point** of φ is a point x^* such $x^* \in \varphi(x^*)$.

THEOREM 8.1 (Brouwer): Let X be a nonempty compact convex subset of R^ℓ and $f : X \to X$ be a continuous function. Then there exists a fixed point of f.

THEOREM 8.2 (Browder): Let X be a nonempty compact convex subset of R^ℓ and a correspondence $\varphi : X \to 2^X$ be nonempty-valued and convex-valued. If φ has open

[3] A Hausdorff space is paracompact if every cover has an open locally finite refinement cover

[4] A Banach space is a normed space that is also a complete metric space under the metric induced by its norm

lower sections, then there exists a fixed point of φ.

THEOREM 8.3: Let X be a nonempty compact convex subset of R^ℓ and a correspondence $\varphi : X \to 2^X$ be nonempty-valued and convex-valued. If φ is lower hemi-continuous, then there exists a fixed point of φ.

THEOREM 8.4 (Kakutani): Let X be a nonempty compact convex subset of R^ℓ and the correspondence $\varphi : X \to 2^X$ be nonempty-valued and convex-valued. If φ has closed graph (or is closed-valued and upper hemi-continuous), then there exists a fixed point of φ.

N. B. Note that Theorems 6.1 is a corollary of Theorem 6.4, and that Theorem 6.2 is a corollary of Theorem 6.3.

9 Probability

DEFINITION 9.1: Ω is a **state space** (set of states of nature). $A \subset \Omega$ is an **event**.

DEFINITION 9.2: A family $\mathcal{F}$ of subsets of Ω is a σ-**algebra** if

(1) $\Omega \in \mathcal{F}$,

(2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,

(3) $A_n \in \mathcal{F}, \forall n \in \mathbf{N} \Rightarrow \bigcup_{n \in N} A_n \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a **measurable space**.

DEFINITION 9.3 : Let $\mathcal{A}$ be a family of subsets of Ω. We denote by $\sigma(\mathcal{A})$ the smallest σ-field containing $\mathcal{A}$.

DEFINITION 9.4: For a topological space (X, τ) , $\mathcal{B}(X) := \sigma(\tau)$ is the **Borel σ-field** on X.

Example

(1) 2^Ω is a σ-field.

(2) $\Omega = \{\omega_1, \omega_2, \omega_3\}$. $\mathcal{F} = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \emptyset, \Omega\}$ is a σ-field.

(3) $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$.
$\mathcal{F} = \{\{\omega_1\}, \{\omega_5\}, \{\omega_1, \omega_5\}, \{\omega_2, \omega_3, \omega_4\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_2, \omega_3, \omega_4, \omega_5\}, \emptyset, \Omega\}$ is a σ-field.

DEFINITION 9.5 : Let $\mathcal{F}_1, \mathcal{F}_2$ be σ-fields on Ω.

(1) $\mathcal{F}_1$ is **finer** than $\mathcal{F}_2$ and, $\mathcal{F}_2$ is **coarser** than $\mathcal{F}_1$ if $\mathcal{F}_2 \subset \mathcal{F}_1$.

(2) The **join** $\mathcal{F}_1 \vee \mathcal{F}_2$ of $\mathcal{F}_1$ and $\mathcal{F}_2$ is the smallest σ-field containing both $\mathcal{F}_1$ and $\mathcal{F}_2$.

(3) The **meet** $\mathcal{F}_1 \wedge \mathcal{F}_2$ of $\mathcal{F}_1$ and $\mathcal{F}_2$ is the largest σ-field contained in both $\mathcal{F}_1$ and $\mathcal{F}_2$.

Example :

$$\begin{aligned} \Omega &= \{\omega_1, \omega_2, \omega_3\}, \\ \mathcal{F}_1 &= \{\{\omega_1, \omega_2\}, \{\omega_3\}, \emptyset, \Omega\}, \\ \mathcal{F}_2 &= \{\{\omega_1, \omega_3\}, \{\omega_2\}, \emptyset, \Omega\}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{F}_1 \vee \mathcal{F}_2 &= \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}, \emptyset, \Omega\}, \\ \mathcal{F}_1 \wedge \mathcal{F}_2 &= \{\emptyset, \Omega\}. \end{aligned}$$

DEFINITION 9.6: A **finite partition** of Ω is a finite family of disjoint subsets of Ω, whose union is Ω.

DEFINITION 9.7: A partition $\mathcal{F}'$ of Ω is a **measurable partition** of Ω if $\mathcal{F}' \subset \mathcal{F}$.
N. B. An information of an agent can be described by a measurable partition of Ω.

DEFINITION 9.8: Let $(\Omega, \mathcal{F})$ be a measurable space. A mapping $\mu : \mathcal{F} \mapsto \Re_+$ is a **measure** if

(1) $\mu(\emptyset) = 0$,

(2) $A_n \in \mathcal{F}, \forall i \in \mathcal{N}$ with $A_i \cap A_j = \emptyset, \forall i \neq j \Rightarrow \mu(\cup_{n \in N} A_n) = \sum_{n \in N} \mu(A_n)$.

$(\Omega, \mathcal{F}, \mu)$ is called a **measure space**.

NOTE μ is a **probability measure** with additional condition $\mu(\Omega) = 1$, and $(\Omega, \mathcal{F}, \mu)$ is called a **probability space**.

DEFINITION 9.9: Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces. $f : \Omega \to \Omega'$ is $(\mathcal{F}, \mathcal{F}')$ **-measurable** if $f^{-1}(A) \in \mathcal{F}, \forall A \in \mathcal{F}'$.

DEFINITION 9.10: Let $f : \Omega \to R$. f is **measurable** with respect to $\mathcal{F}$ (or $\mathcal{F}$-measurable) if $f^{-1}(A) \in \mathcal{F}, \forall A \subset \mathcal{B}(R)$.

NOTE A **random variable** is a real-valued measurable function in probability space.

DEFINITION 9.11: Let X_i's be random variables. $\sigma(X_1, \ldots, X_n)$ is the smallest σ-field with respect to which $X_1, \ldots, X_n$ are measuarble.

N. B. $\sigma(X_1, \ldots, X_n) = \bigvee_{i=1}^n \sigma(X_i)$.

Example : Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\mathcal{F} = 2^\Omega$. A consumer has a random endowments : $e(\omega_1) = 1, e(\omega_2) = 0, e(\omega_3) = 0$. Then e is measurable with respect to $\sigma(e) = \{\{\omega_1\}, \{\omega_2, \omega_3\}, \Omega, \emptyset\}$

DEFINITION 9.12: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let f be a nonnegative measurable simple function, i.e., $f = \sum_{i=1}^n a_n \mathbf{1}_{A_i}$ where $a_i \in \mathbf{R}_+, \forall i = 1, \ldots, n$ and $(A_1, \ldots, A_n)$ be a finite measurable partition of Ω. The **integral** of f on Ω is defined by

$$\int f d\mu = \sum_{i=1}^n a_i \mu(A_i)$$

DEFINITION 9.13: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. The **integral** of a nonnegative random variable f on Ω is defined by

$$\int f d\mu = \sup\{\int f' d\mu : f' \text{ is simple and } f' \leq f\}$$

DEFINITION 9.14: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $A_1, \ldots, A_n$ be a finite measurable partition Ω. The **integral** of a random variable f on Ω is defined by

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu,$$

where $f = f^+ - f^-$ and $f^+ = f \vee 0, f^- = (-f) \vee 0$.

Monotone Convergence Theorem : Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and (f_n) be a sequence of measurable functions on Ω.

$$0 \leq f_n \leq f_{n+1}, \forall n \in \mathbf{N} \text{ and } f_n \to f, \mu\text{-}a.e. \Rightarrow \int f_n d\mu \to \int f d\mu.$$

Lemma (Fatou) : Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and (f_n) be a sequence of measurable functions on Ω.

$$\int \liminf f_n d\mu \leq \liminf \int f_n d\mu$$

Dominated Convergence Theorem : Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and (f_n) be a sequence of measurable functions on Ω. Suppose that g is a nonnegative integrable function on Ω and f is a measurable function Ω such that

$$|f_n| \leq g, \forall n \in N \text{ and } f_n \to f, \mu\text{-}a.e..$$

Then f and $f_n, n \in \mathcal{N}$ are integrable and $\int f_n d\mu \to \int f d\mu$.

10 Information Structure

We have a **probability measure space** $(\Omega, \mathcal{F}, \mu)$. There are m agents and μ is their **common prior**.

DEFINITION 10.1: An **information correspondence** is a nonempty-valued correspondence $P : \Omega \to \mathcal{F}$.

The interpretation is that when the state is ω the decision-maker knows only that the state is in $P(\omega)$. When we use information correspondence to model a decision-maker's knowledge we usually assume the following two conditions:

(i) For every $\omega \in \Omega$, $\omega \in P(\omega)$,

(ii) If $\omega' \in P(\omega)$, then $P(\omega') = P(\omega)$.

(i) says that the decision maker never excludes the true state from the set of states he regards as feasible. (ii) says that the decision-maker uses the consistency or inconsistency of states with his information to make inferences about the state.

DEFINITION 10.2: An information correspondence P is an **information partition** if there is a partition $\mathcal{P}$ such that for every $\omega \in \Omega$, $\omega \in P(\omega) \in \mathcal{P}$. The partition $\mathcal{P}$ is said to be **generated by** P.

N. B. An information partition P can be identified with the partition that it generates.

Lemma An information correspondence P is an information partition if and only if it satisfies (i) and (ii).

The **(private) information** of agent i is represented by an information partition P_i on Ω. The **information set** of agent i at ω is given by $P_i(\omega)$.

Definition :

(1) An information P' is **finer** than an information P and P is **coarser** than P' if $P'(\omega) \subset P(\omega)$, $\forall\, \omega \in \Omega$.

(2) A **meet**, $\bigwedge_{i \in S} P_i$, of $\{P_i : i \in S\}$ is the finest information that is coarser than P_i for every $i \in S$.

(3) A **join**, $\bigvee_{i \in S} P_i$, of $\{P_i : i \in S\}$ is the coarsest information that is finer than P_i for every $i \in S$.

N. B. Note that P' is finer than P iff $\sigma(P) \subset \sigma(P')$ where $\sigma(P)$ is the smallest σ-field containing the partition generated by P.

N. B. The finest information is given by P such that $P(\omega) = \{\omega\}$ for every $\omega \in \Omega$. The coarsest information is given by P such that $P(\omega) = \Omega$ for every ω.

THEOREM 10.1: The following hold :

(1) For every $\omega \in \Omega$, $(\bigwedge_{i \in S} P_i)(\omega) = \bigcup_{i \in S} \{P_i(\omega') : \omega' \in (\bigwedge_{i \in S} P_i)(\omega)\}$,

(2) $\sigma(\bigwedge_{i \in S} P_i) = \bigcap_{i \in S} \sigma(P_i)$,

(3) For every $\omega \in \Omega$, $(\bigvee_{i \in S} P_i)(\omega) = \bigcap_{i \in S} [P_i(\omega)]$,

(4) $\sigma(\bigvee_{i \in S} P_i) = \sigma(\bigcup_{i \in S} P_i)$.

THEOREM 10.2: If P' is finer than P, then $P' \wedge P = P$ and $P' \vee P = P'$.

Given our interpretation of an information correspondence, a decision-maker for whom $P(\omega) \subset A$ knows, in the state ω, that some state in the event A has occurred. In this case we say that in the state ω the decision-maker knows A. For every $\omega \in \Omega$, agent i knows that $P_i(\omega)$ occurs at ω because of property (i).

DEFINITION 10.3: An event A **occurs at** ω if $\omega \in A$.

DEFINITION 10.4: Agent i **knows that** A **occurs at** ω if $P_i(\omega) \subset A$. **N. B.** For every $\omega \in \Omega$, agent i knows that $P_i(\omega)$ occurs at ω.

DEFINITION 10.5: An event A is **common knowledge of** S **at** ω if $(\bigwedge_{i \in S} P_i)(\omega) \subset A$.

In the game theory context, common knowledge is defined by different, but equivalent way. For more, refer to chapter 5 of Osborne-Rubinstein or chapter 14 of Fudenberg-Tirole.

DEFINITION 10.6: A function $x : \Omega \to R$ is P_i**-measurable** if $x^{-1}(B) \in \sigma(P_i)$ for every $B \in \mathcal{B}(R)$.

DEFINITION 10.7: A partition $P(x_1, \ldots, x_n)$ is the smallest partition with respect to which $x_1, \ldots, x_n$ are measurable. It is said to be **generated by** $x_1, \ldots, x_n$.

THEOREM 10.3 Let x_i be P_i-measurable for every i.

(1) If P_i' is finer than P_i, then x_i is P_i'-measurable.

(2) If $\lambda \in R$, then λx_i is P_i-measurable.

(3) $\sum_{i \in S} x_i$ is $(\bigvee_{i \in S} P_i)$-measurable, where $(\sum_{i \in S} x_i)(\omega) = \sum_{i \in S} x_i(\omega)$.

(4) $(x_i)_{i \in S}$ is $(\bigvee_{i \in S} P_i)$-measurable, where $\left((x_i)_{i \in S}\right)(\omega) = (x_i(\omega))_{i \in S}$.

(5) $\prod_{i \in S} x_i$ is $(\bigvee_{i \in S} P_i)$-measurable, where $(\prod_{i \in S} x_i)(\omega) = \prod_{i \in S} x_i(\omega)$.

Example

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_5\}$ and $\mathcal{F} = 2^{\Omega}$. Consider three agents whose information is given by

$$
\begin{aligned}
P_1(\Omega) &= \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}, \{\omega_5\}\}, \\
P_2(\Omega) &= \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5\}\}, \\
P_3(\Omega) &= \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}\}
\end{aligned}
$$

We will use P_i instead of $P_i(\Omega)$ by abusing of notation.

1. The information set of agent 2 at ω_3 is $P_2(\omega_3) = \{\omega_3, \omega_4\}$.

2. An event $A_1 = \{\omega_1, \omega_2, \omega_3\}$ occurs at ω_2.

3. Agent 1 knows that A_1 occurs at ω_1, and knows that $P_1(\omega_2) = \{\omega_2, \omega_3\}$ occurs at ω_2.

4. P_3 is finer than P_2 and P_2 is coarser than P_3.

5 $\sigma(P_2) = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_1, \omega_2, \omega_5\}, \{\omega_3, \omega_4, \omega_5\}, \emptyset, \Omega\}$.

6. We can show that

$$\begin{aligned}
P_1 \wedge P_2 &= \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5\}\}, \\
P_1 \wedge P_3 &= \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}, \{\omega_5\}\}, \\
P_2 \wedge P_3 &= P_2.
\end{aligned}$$

Note that $P_1 \wedge P_2$ is coarser than P_1.

7. We can show that

$$\begin{aligned}
P_1 \vee P_2 &= \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}\}, \\
P_1 \vee P_3 &= P_1 \vee P_2, \\
P_2 \vee P_3 &= P_3.
\end{aligned}$$

Note that $P_1 \vee P_2$ is finer than P_1.

8. We can show that

$$\bigwedge_{i=1}^{3} P_i = P_1 \wedge P_2, \quad \bigvee_{i=1}^{3} P_i = P_1 \vee P_2.$$

9. $A_2 = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ is common knowledge of $\{1, 2, 3\}$ at ω_2 and $A_3 = \{\omega_5\}$ is commom knowledge of $\{1, 2, 3\}$ at ω_5, but $A_1 = \{\omega_1, \omega_2, \omega_3\}$ is not common knowledge of $\{1, 2, 3\}$ at any $\omega \in \Omega$. Note that at ω_2, every agent knows that A_1 occurs. However, A_1 is common knowledge of $\{1, 3\}$ at ω_2.

10. Let us write $x = (x(\omega_k))_{k=1}^{5}$. Consider the following random variables $x_1, \ldots, x_8$:

$$\begin{aligned}
x_1 &= (1, 2, 3, 4, 5), \quad x_2 = (0, 1, 1, 4, 2), \\
x_3 &= (1, 1, 3, 3, 2), \quad x_4 = (0, 0, 2, 4, 5), \\
x_5 &= (1, 1, 1, 1, 1), \quad x_6 = (1, 1, 1, 1, 5), \\
x_7 &= (1, 0, 0, 3, 1), \quad x_8 = (0, 0, 0, 1, 2);
\end{aligned}$$

(1) We can show that

$$\begin{aligned}
P(x_8) &= \{\{\omega_1,\omega_2,\omega_3\},\{\omega_4\},\{\omega_5\}\}, \\
P(x_2,x_3) &= \{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\},\{\omega_5\}\} = P(x_2)\vee P(x_3), \\
\sigma(x_7) &= \{\{\omega_1,\omega_5\},\{\omega_2,\omega_3\},\{\omega_4\},\{\omega_1,\omega_4,\omega_5\},\{\omega_2,\omega_3,\omega_4\},\{\omega_1,\omega_2,\omega_3,\omega_5\},\emptyset,\Omega\}, \\
\sigma(x_3,x_6) &= \{\{\omega_1,\omega_2\},\{\omega_3,\omega_4\},\{\omega_5\},\{\omega_1,\omega_2,\omega_5\},\{\omega_3,\omega_4,\omega_5\},\{\omega_1,\omega_2,\omega_3,\omega_4\},\emptyset,\Omega\} \\
&= \sigma(x_3)\vee\sigma(x_6).
\end{aligned}$$

(2) x_1 and x_7 are $(P_1 \vee P_2)$-measurable, x_6 is $(P_1 \wedge P_2)$-measurable, x_8 is $(P_1 \wedge P_3)$-measurable and x_5 is P_i-measurable for every i.

(3) Since x_3 is P_2-measurable and P_3 is finer than P_2, x_3 is also P_3-measurable.

(4) Since x_2 is P_1-measurable, $3x_2 = (0,3,3,12,6)$ is still P_1-measurable.

(5) $x_2 - 2x_3 = (-2,-1,-5,-2,-2)$ is $(P_1 \vee P_2)$-measurable.

(6) $(x_3,x_4) = ((1,0),(1,0),(3,2),(3,4),(2,5))$ is $(P_2 \vee P_3)$-measurable.

(7) $x_2x_4 = (0,0,2,16,10)$ is $(P_1 \vee P_3)$-measurable.

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Part II

GENERAL EQUILIBRIUM

1 Walrasian Equilibrium

1.1 Preferences

DEFINITION 1.1.1 A **relation** R is a correspondence from X to 2^X. The properties of R are defined as follows.

(1) R is **reflexive** if, for every $x \in X$, $x \in R(x)$.

(2) R is **irreflexive** if, for every $x \in X$, $x \notin R(x)$.

(3) R is **complete** if, for every $x, x' \in X$, $x' \in R(x)$ or $x \in R(x')$.

(4) R is **transitive** if $x'' \in R(x')$ and $x' \in R(x)$ implies $x'' \in R(x)$.

(5) R is **negatively transitive** if $x'' \notin R(x')$ and $x' \notin R(x)$ implies $x'' \notin R(x)$.

(6) R is **symmetric** if $x' \in R(x)$ implies $x \in R(x')$.

(7) R is **asymmetric** if $x' \in R(x)$ implies $x \notin R(x')$.

(8) R is **antisymmetric** if $x' \in R(x)$ and $x \in R(x')$ implies $x' = x$.

THEOREM 1.1.1 Let $P : X \to 2^X$ be a relation.

(1) If P is asymmetric, then it is irreflexive.

(2) If P is asymmetric and negatively transitive, then it is transitive.

THEOREM 1.1.2 Define relations $R : X \to 2^X$ and $I : X \to 2^X$ by $R(x) := \{x' \in X : x \notin P(x')\}$ and $I(x) = \{x' \in X : x' \in R(x) \text{ and } x \in R(x')\}$. Then

(1) P is asymmetric iff R is complete.

(2) P is negatively transitive iff R is transitive.

(3) P is asymmetric and negatively transitive implies that I is reflexive, symmetric, and transitive.

N. B Note that $R(x) = X \setminus P^{-1}(x)$ and $I(x) = R(x) \cap R^{-1}(x)$.

DEFINITION 1.1.2 Define a relation $R : X \to 2^X$ by $R(x) = \{x' \in X : x' \succeq x\}$. The properties of $\succeq$ (or R) are defined as follows. [5]

(1) $\succeq$ is **reflexive** if, for every $x \in X$, $x \succeq x$.

(2) $\succeq$ is **complete** if, for every $x, x' \in X$, $x' \succeq x$ or $x \succeq x'$.

(3) $\succeq$ is **transitive** if $x \succeq x'$ and $x' \succeq x''$ implies that $x \succeq x''$.

(4) $\succeq$ is **weakly monotonic** if $x' \geq x$ implies $x' \succeq x$.

(5) $\succeq$ is **monotonic** if $x' \gg x$ implies $x' \succ x$.

(6) $\succeq$ is **strongly monotonic** if $x' > x$ implies $x' \succ x$.

(7) $\succeq$ is **nonsatiated** if, for every $x \in X$, there is $x' \in X$ such that $x' \succ x$.

(8) $\succeq$ is **locally nonsatiated** if, for every $x \in X$, for every $\varepsilon > 0$, there is a $x' \in B_\varepsilon(x) \cap X$ such that $x' \succ x$.

(9) $\succeq$ is **convex** if $x' \succeq x$ and $x'' \succeq x$ implies $\alpha x' + (1-\alpha)x'' \succeq x$ for every $\alpha \in [0,1]$.

(10) $\succeq$ is **semi-strictly convex** if $x' \succ x$ implies $\alpha x' + (1-\alpha)x \succ x$ for $\alpha \in (0,1]$ and $x' \sim x$ implies $\alpha x' + (1-\alpha)x \succeq x$ for every $\alpha \in [0,1]$.

(11) $\succeq$ is **strictly convex** if $x' \succeq x$ and $x'' \succeq x$ and $x' \neq x''$ implies $\alpha x' + (1-\alpha)x'' \succ x$ for every $\alpha \in (0,1)$.

(12) $\succeq$ has an **extremely desirable bundle** v if for every $x \in X$ and for every $\alpha \in R_+$, $x + \alpha v \in \{x' \in X : x' \succ x\}$.

---

[5] Define a relation $P : X \to 2^X$ by $P(x) = X \setminus R^{-1}(x) = \{x' \in X : x' \succ x\}$.

(1) $\succ$ is **irreflexive** if $x \notin P(x)$.

(2) $\succ$ is **transitive** if $x \in P(y)$ and $y \in P(z)$ implies $x \in P(z)$.

(3) $\succ$ is **continuous** if $P(x)$ and $P^{-1}(x)$ are open for every $x \in X$.

(4) $\succ$ is **monotonic** if $x' \gg x$ implies $x' \in P(x)$.

(5) $\succ$ is **strictly monotonic** if $x' > x$ implies $x' \in P(x)$.

(6) $\succ$ is **convex** if P is convex-valued.

(7) $\succ$ is **strictly convex** if $x \neq x'$ implies that $\alpha x + (1-\alpha)x' \in P(x) \cup P(x')$ for every $\alpha \in (0,1)$.

(13) $\succeq$ is **proper** at x if there exists a $v \in R^\ell_+ \setminus \{0\}$ and a neighborhood V of zero such that $z \in R^\ell$ and $x - \alpha v + z \succeq x$ with $\alpha \in R_+$ implies $z \notin \alpha V$.

(14) $\succeq$ is **uniformly proper** if it is proper at every $x \in X$.

N. B We can define the convexities of preference $\succeq$ in the following forms.

(i) $x' \succeq x$ implies $\alpha x' + (1-\alpha)x \succeq x$ for every $\alpha \in [0,1]$.

(ii) $x' \succ x$ implies $\alpha x' + (1-\alpha)x \succ x$ for every $\alpha \in (0,1]$.

(iii) $x' \sim x$ with $x' \neq x$ implies $\alpha x' + (1-\alpha)x \succ x$ for every $\alpha \in (0,1)$.

THEOREM 1.1.3 We assume that $\succeq$ is complete and transitive.

(1) If $\succeq$ continuous. Then (iii) implies (ii), which,in turn, implies (i).

(2) (i) holds for every $x \in X$ iff $\succeq$ is convex.

(3) $\succeq$ is convex iff $\succ$ is convex.

(4) If $\succeq$ is convex, continuous, strictly monotone, then (ii) holds.

(5) If $\succeq$ is continuous, then (iii) holds iff $\succeq$ is strictly convex.

THEOREM 1.1.4 A preference $\succeq$ is proper iff there is a non-trivial open cone $\Gamma \in R^\ell$ such that for every $x \in X$,

$$\Gamma \cap (-R^\ell_+) \neq \emptyset, \ (\{x_i\} + \Gamma) \cap R(x_i) = \emptyset.$$

N. B There is a commodity which is very desirable in the sense that its marginal rates of substitution with respect to any other commodity are uniformly bounded above. It is always satisfied by monotone preference. It can be equivalently formulated as : there is a non-trivial open cone Γ such that

$$(\{x_i\} + \Gamma) \cap X_i \subset P(x_i)$$

for every i and $x_i \in X_i$.

THEOREM 1.1.5 Every uniformly proper vector is extremely desirable.

THEOREM 1.1.6 If a preference $\succeq$ is monotonic and has extremely desirable bundle, then it is uniformly proper.

1.2 Gale-Debreu-Nikaido Lemma

THEOREM 1.2.1 (Gale-Debreu-Nikaido) : Let $Z : \Delta \to 2^{R^\ell}$ be an excess demand correspondence satisfying the following conditions:

(1) Z is nonempty-valued, compact-valued, convex-valued, and upper hemi-continuous,

(2) for every $p \in \Delta, \exists z \in Z(p)$ such that $p \cdot z \leq 0$.

Then, $\exists p^* \in \Delta$ such that $Z(p^*) \cap R^\ell_- \neq \emptyset$.

PROOF: Suppose otherwise, i.e., $\forall p \in \Delta$, $Z(p) \cap R^\ell_- = \emptyset$. Since $Z(p)$ is nonempty compact convex and R^ℓ_- is nonempty closed convex, by the separating hyperplane theorem, $\exists q^* \in R^\ell \setminus \{0\}$ such that $\sup_{y \in R^\ell_-} q^* \cdot y < \inf_{z \in Z(p)} q^* \cdot z$. Note that $\sup_{y \in R^\ell_-} q^* \cdot y = 0$ so that $q^* \cdot z > 0, \forall z \in Z(p)$. Without loss of generality, we can take $q^* \in \Delta$.
Define $F : \Delta \to 2^\Delta$ by $F(p) := \{q \in \Delta : q \cdot z > 0, \forall z \in Z(p)\}$. Then, we want to show that F is nonempty, convex valued and lower-hemicontinuous, so that we can apply the Michael selection theorem.
Since for every $p \in \Delta$, $q^* \cdot z > 0, \forall z \in Z(p)$, $q^* \in F(p)$, i.e., F is nonempty valued. Pick q_1, q_2 in $F(p)$. Then for every $p \in \Delta, q_1 \cdot z > 0$ and $q_2 \cdot z > 0, \forall z \in Z(p)$. Thus for every $p \in \Delta$ and for every $\alpha \in [0,1]$, $(\alpha q_1 + (1-\alpha) q_2) \cdot z > 0, \forall z \in Z(p)$, which implies that $\alpha q_1 + (1-\alpha) q_2 \in F(p)$, i.e., F is convex valued.
For each $q \in \Delta$,

$$
\begin{aligned}
F^{-1}(q) &= \{p \in \Delta : q \in F(p)\} && (1)\\
&= \{p \in \Delta : q \cdot z > 0, \forall z \in Z(p)\} && (2)\\
&= \{p \in \Delta : Z(p) \subset \{z : q \cdot z > 0\}\}. && (3)
\end{aligned}
$$

Since $V := \{z : q \cdot z > 0\}$ is open and Z is upper-hemicontinuous, $\forall q \in \Delta, F^{-1}(q)$ is open in Δ, *i.e.*, F has open lower sections. Thus, F is lower-hemicontinuous[6].
By the Michael selection theorem, there exists a continuous function $f : \Delta \to \Delta$ such that $f(p) \in F(p)$ for all $p \in \Delta$. This function maps points from the nonempty compact convex set into itself. Therefore, it fulfills the condition of the Brouwer's fixed point theorem.

[6]Note that for every open subset W of Δ, $\bigcup_{q \in W} F^{-1}(q) = \{p \in \Delta : F(p) \cap W \neq \emptyset\}$. In fact, $p' \in \cup_{q \in W} F^{-1}(q)$ iff $p' \in F^{-1}(q)$ for some $q \in W$ iff $q \in F(p'), q \in W$ iff $q \in F(p') \cap W$ iff $F(p') \cap W \neq \emptyset$ iff $p' \in \{p \in \Delta : F(p) \cap W \neq \emptyset\}$. Since the union of open sets is open, $\{p \in \Delta : F(p) \cap W \neq \emptyset\}$ is open for every open subset W of Δ.

By the Brouwer's fixed point theorem, there exists a $p^* = f(p^*) \in F(p^*)$ Hence, $p^* \cdot z > 0, \forall z \in Z(p^*)$. This contradicts the condition (2), which is the Walras law. $\square$

ALTERNATIVE PROOF: (by applying KKM Theorem)
Define $F : \Delta \to 2^{\Delta}$ by $F(p) = \{q \in \Delta : q \cdot z > 0, \forall z \in Z(p)\}$. Then F is convex valued and has open lower sections. Since $F^{-1}(q)$ is open $\forall q \in \Delta$, $G(q) = \Delta \backslash F^{-1}(q)$ is closed $\forall q \in \Delta$. Need to show that $G(q)$ satisfies conditions of KKM theorem.
i) For any set of points $\{q_1, \ldots, q_n\} \subset \Delta$, $co\{q_i, \ldots, q_n\} \subset \bigcup_{i=1}^n G(q_i)$.
Suppose not. Let $p \in co\{q_i, \ldots, q_n\}$ and $p \notin \bigcup_{i=1}^n G(q_i) \Rightarrow p \notin G(q_i), \forall i \Rightarrow p \in F^{-1}(q_i), \forall i$ $\Rightarrow q_i \in F(p), \forall i \Rightarrow co\{q_i, \ldots, q_n\} \subset coF(p) = F(p) \Rightarrow p \in F(p)$, which is a contradiction to condition (2).
ii) $G(q)$ is compact for each $q \in \Delta$, since it is a closed subset of a compact set Δ.
Therefore, by applying KKM theorem, $\bigcap_{q \in \Delta} G(q) \neq \emptyset$.
Let $p \in \bigcap_{q \in \Delta} G(q) \Rightarrow p \in G(q), \forall q \in \Delta \Rightarrow p \notin F^{-1}(q), \forall q \in \Delta \Rightarrow q \notin F(p), \forall q \in \Delta$ $\Rightarrow F(p) = \emptyset$, for some $p \in \Delta$
But $F(p) = \emptyset$ for some $p \in \Delta$ implies that $\forall q \in \Delta, q \cdot z \leq 0$ for some $z \in Z(p)$, which in turn implies that $Z(p) \cap R^{\ell}_{-} \neq \emptyset$.
To see this, suppose otherwise, i.e., $Z(p) \cap R^{\ell}_{-} = \emptyset$. Since $Z(p)$ is nonempty, compact, convex and R^{ℓ}_{-} is nonempty closed convex, by the separating hyperplane theorem, $\exists q^* \in R^{\ell} \setminus \{0\}$ such that $\sup_{y \in R^{\ell}_{-}} q^* \cdot y < \inf_{z \in Z(p)} q^* \cdot z$. Note that $\sup_{y \in R^{\ell}_{-}} q^* \cdot y = 0$ so that $q^* \cdot z > 0, \forall z \in Z(p)$, a contradiction. Hence for some $p \in \Delta$, $Z(p) \cap R^{\ell}_{-} \neq \emptyset$, as it was to be shown. $\square$

1.3 Existence of Walrasian Equilibrium

THEOREM 1.3.1 : Let $\mathcal{E} = \{(X_i, u_i, e_i) : i \in I\}$ be an exchange economy satisfying the following assumptions for each $i \in I$.

(a) X_i : is a nonempty, compact, convex subset of R^{ℓ},

(b) $u_i : X_i \to R_+$ is quasi-concave and continuous,

(c) $e_i \in int X_i$.

Then $\mathcal{E}$ has a free disposal equilibrium, i.e., there exist $(p^*, x^*) \in \Delta \times X$ with $X = \prod_{i \in I} X_i$ such that

(1) $\forall i \in I, x_i^* \in \varphi_i(p^*) := \{x_i \in \mathcal{B}_i(p^*) : u_i(x_i) \geq u_i(x_i'), \forall x_i' \in \mathcal{B}_i(p^*)\}$, where $\mathcal{B}_i(p^*) := \{x_i \in X_i : p^* \cdot x_i \leq p^* \cdot e_i\}$,

(2) $\sum_{i\in I} x_i^* \leq \sum_{i\in I} e_i$

PROOF: Since $\forall p \in \Delta$, $e_i \in \mathcal{B}_i(p)$, $\mathcal{B}_i$ is nonempty-valued. Clearly, $\mathcal{B}_i$ is closed-valued. Since a closed subset of a compact set is compact, $\mathcal{B}_i$ is compact-valued. It is easy to verify that $\mathcal{B}_i$ is convex-valued. Finally, $\mathcal{B}_i$ is continuous. Let $v_i(x,p) := u_i(x)$. By the Maximum Theorem, the demand correspondence φ_i is nonempty-valued, compact-valued and upper hemicontinuous. Furthermore, the quasi-concavity of the utility function and the convex-valuedness of $\mathcal{B}_i$ implies that φ_i is convex-valued. Define the excess demand correspondence $Z : \Delta \to 2^{R^\ell}$ by $Z(p) := \sum_{i\in I} \varphi_i(p) - \sum_i e_i$. Then Z is nonempty compact convex valued and upper-hemicontinuous. Moreover, for every $i \in I$, $\forall p \in \Delta, p \cdot x_i \leq p \cdot e_i$ so that $p \cdot z \leq 0, \forall z \in Z(p)$. By the DGN lemma, $\exists p^* \in \Delta$ such that $Z(p^*) \cap R^\ell_- \neq \emptyset$. Take $z^* \in Z(p^*) \cap R^\ell_-$. Then for every i, there exists $x_i^* \in \varphi_i(p^*)$ such that $\sum_i x_i^* - \sum_i e_i = z^* \leq 0$. Hence (p^*, x^*) constitutes a free disposal equilibrium. $\square$

LEMMA 1.3.2 : If $e_i \in intX_i$, then $\mathcal{B}_i$ is lower hemi-continuous.

1.4 Equilibrium in an Abstract Economy

DEFINITION 1.4.1 : A **game (in a normal form)** $\Gamma = \{(X_i, P_i)_i : i \in I\}$ is a set of pairs (X_i, P_i), where

(1) X_i is the strategy set of player i,

(2) $P_i : X \to 2^{X_i}$ is the preference correspondence player i.

DEFINITION 1.4.2 : $x^* \in X$ is a **Nash equilibrium** if for every $i \in I$, $P_i(x^*) := \{y_i \in X_i : (x_1^*, \ldots, y_i, \ldots, x_n^*) \succ_i (x_1^*, \ldots, x_i^*, \ldots, x_n^*)\} = \emptyset$.

DEFINITION 1.4.3 : An **abstract economy** Γ is a set of triplets $\{(X_i, P_i, A_i) : i \in I\}$ where

(1) X_i is the strategy set of agent i,

(2) $P_i : X \to 2^{X_i}$ is the preference correspondence of agent i,

(3) $A_i : X \to 2^{X_i}$ is the constraint correspondence of agent i.

DEFINITION 1.4.4 : An **equilibrium** for the abstract economy Γ is $x^* \in X$ such that, for every $i \in I$,

(1) $x_i^* \in A_i(x^*)$,

(2) $P_i(x^*) \cap A_i(x^*) = \emptyset$.

THEOREM 1.4.1 : Let $\Gamma = \{(X_i, P_i, A_i) : i \in I\}$ be an abstract economy satisfying the following assumptions for every $i \in I$.

(1) X_i is a nonempty, compact, convex subset of R^ℓ,

(2) P_i has an open graph in $X \times X_i$,

(3) $x_i \notin coP_i(x), \forall x \in X$,

(4) $A_i : X \to 2^{X_i}$ is nonempty, closed, convex valued and continuous correspondence.

Then Γ has an equilibrium, i.e., there exists $x^* \in X$ such that for every $i \in I$,

(a) $x_i^* \in A_i(x^*)$,

(b) $P_i(x^*) \cap A_i(x^*) = \emptyset$

PROOF: For each i, define a correspondence $\varphi_i : X \to 2^{X_i}$ by $\varphi_i(x) := coP_i(x)$. Since P_i has open graph, so does φ_i. For each i, define a correspondence $\psi_i : X \to 2^{X_i}$ by $\psi_i(x) := \varphi_i(x) \cap A_i(x)$. Since φ_i has an open graph and A_i is lower-hemicontinuous, it follows that ψ_i is lower-hemicontinuous. Moreover ψ_i is convex-valued. For each i, define $V_i := \{x \in X : \psi_i(x) \neq \emptyset\}$.

(i) If V_i is empty, (b) is satisfied for all $x \in X$. Since A_i is nonempty, closed, convex valued and continuous, so is A, where $A(x) = \prod_{i \in I} A_i(x)$. By the Kakutani fixed point theorem, there exists $x^* \in A(x^*)$ so that $x_i^* \in A_i(x^*)$. Hence x^* is an equilibrium of Γ.

(ii) Suppose V_i is not empty. Since ψ_i is lower-hemicontinuous, $V_i = \{x \in X : \psi_i(x) \cap X_i \neq \emptyset\}$ is open. Let $\psi_i|_{V_i} : V_i \to 2^{X_i}$ be a restriction of ψ_i to V_i. Then it is nonempty convex valued and lower-hemicontinuous. By the Michael selection theorem, there exists a continuous function $f_i : V_i \to X_i$ such that $f_i(x) \in \psi_i|_{V_i}(x)$ for all $x \in V_i$. For each i, define a correspondence $F_i : X \to 2^{X_i}$ as follows.

$$F_i(x) := \begin{cases} \{f_i(x)\} & \text{if } x \in V_i \\ A_i(x) & \text{if } x \notin V_i \end{cases}$$

Then F_i is nonempty closed convex valued and upper-hemicontinuous. Define $\Psi : X \to 2^X$ by $\Psi(x) := \prod_i^n F_i(x)$. Then Ψ is nonempty closed convex valued and upper-hemicontinuous. Therefore, by the Kakutani fixed point theorem, there exists $x^* \in X$ such that $x^* \in \Psi(x^*)$. If for some i, $x^* \in V_i$, it follows from the definition of F_i that $x_i^* = f_i(x^*) \in \psi_i(x^*) \subset coP_i(x^*)$, which is a contradiction to (3). Thus for every i, $x^* \notin V_i$ which implies that $x_i^* \in A_i(x^*)$ and $\psi_i(x^*) = \emptyset$, i.e., $P_i(x^*) \cap A_i(x^*) = \emptyset$. Hence x^* is an equilibrium of Γ. □
Next we use the above Theorem to prove the existence of Nash equilibrium for a game in a normal form as a Corollary.

COROLLARY 1.4.1: Let $\mathcal{G} = \{(X_i, u_i) : i = 1, 2, ..., n\}$ be a **game in normal form** satisfying for all i the following assumptions:
i) X_i is compact, convex and non empty subset of R^l,
ii) $u_i : \Pi_{j=1}^n X_j \to R$ is quasi - concave and continuous.
Then $\mathcal{G}$ has a Nash equilibrium, i.e., there exists an $x^* \in X = \Pi_{i=1}^n X_i$ such that forall i,

$$u_i(x_1^*, ..., x_n^*) \geq u_i(x_1^*, ..., y_i, ..., x_n^*), \forall y_i \in X_i$$

PROOF: $\forall i$ set $A_i(x) = X_i$. Also, $\forall i$ define the correspondence $P_i : X \to 2^{X_i}$ by,

$$P_i(x_1, ..., x_n) = \{y_i \in X_i : u_i(x_1, ..., y_i, ..., x_n) > u_i(x_1, ..., x_n)\}$$

Hence, we have an abstract economy $\Gamma = \{(X_i, P_i, A_i) : i = 1, 2, ..., n\}$. We can easily verify the following: a) P_i has an open graph, b) P_i is convex - valued, c) $x_i \notin P_i(x_1, .., x_n), \forall x \in X$ and d) $A_i : X \to 2^{X_i}$ is non empty, closed - valued, convex - valued and continuous. Thus, Γ has an equilibrium, i.e., there exists an $x^* \in X$ such that,
i) $x_i^* \in A_i(x^*), \forall i$,
ii) $A_i(x^*) \cap P_i(x^*) = \emptyset, \forall i$.
From i) and ii) we can deduce that for all i, $x_i^* \in X_i$ and $P_i(x^*) = \emptyset$. That is,

$$\forall y_i \in X_i, u_i(x_1^*, ..., y_i, ..., x_n^*) \leq u_i(x_1^*, ..., x_n^*)$$

□
Now we provide an alternative proof of the existence of a Nash equilibrium in a normal form game (Corollary 1.4.1), by using the Berge Maximum Theorem and the Kakutani Fixed point Theorem.

PROOF: Let $X = \Pi_{i=1}^n X_i$ and $\tilde{X}_i = \Pi_{i \neq j} X_j$. For all i define $\varphi_i : \tilde{X}_i \to 2^{X_i}$ by,

$$\varphi_i(\tilde{x}_i) = \{y_i \in X_i : u_i(y_i, \tilde{x}_i) = \max_{z_i \in X_i} u_i(z_i, \tilde{x}_i)\}$$

By the Berge Maximum Theorem, for all i, φ_i is u.h.c and compact-valued. Also φ_i is nonempty since u_i is a continuous function defined on a compact set (Weierstrass Theorem). Moreover, for all i, φ_i, is convex-valued, since u_i is quasi-concave. Now define a new correspondence $\Phi : X \to 2^X$ by,

$$\Phi(x) = \Pi_{i=1}^{n} \varphi_i(\tilde{x}_i)$$

Notice now, that Φ carries all the properties of φ_i. Hence, Φ has a closed graph (since it is uhc and closed-valued and X is compact), is nonempty and convex-valued. Therefore, by the Kakutani fixed point theorem, there exists an $x^* \in X$ such that $x^* \in \Phi(x^*)$. It can be easily seen that the fixed point by construction is a Nash equilibrium. □

DEFINITION 1.4.5 : An **exchange economy** $\mathcal{E}$ is $\{(X_i, P_i, e_i) : i \in I\}$ where, for every $i \in I$,

(1) X_i is the consumption set of agent i,

(2) $P_i : X \to 2^{X_i}$ is the preference correspondence of agent i,

(3) $e_i \in X_i$ is the initial endowment of i.

DEFINITION 1.4.6 : An **equilibrium for the exchange economy** $\mathcal{E}$ is $(p^*, x^*) \in \Delta \times X$ such that

(a) $\forall i \in I, p^* \cdot x_i^* \leq p^* \cdot e_i$,

(b) $\forall i \in I, P_i(x^*) \cap \{x_i \in X_i : p^* \cdot x_i \leq p^* \cdot e_i\} = \emptyset$,

(c) $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$.

THEOREM 1.4.3 : Let $\mathcal{E} = \{(X_i, P_i, e_i) : i \in I\}$ be an exchange economy satisfying the following assumptions for each $i \in I$.

(1) X_i : is a nonempty compact convex subset of R^ℓ,

(2) P_i has an open graph in $X \times X_i$ and $x_i \notin coP_i(x), \forall x \in X$,

(3) $e_i \in intX_i$.

Then $\mathcal{E}$ has a free disposal equilibrium, i.e., there exist $(p^*, x^*) \in \Delta \times X$ such that

(a) $\forall i \in I, p^* \cdot x_i^* \leq p^* \cdot e_i$,

(b) $\forall i \in I, P_i(x^*) \cap \{x_i \in X_i : p^* \cdot x_i \leq p^* \cdot e_i\} = \emptyset$,

(c) $\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i$.

PROOF: Define $\bar{P}_{n+1} : \Delta \times X \to 2^{\Delta}$ by $\bar{P}_{n+1}(p, x) := \{q \in \Delta : q \cdot (\sum_{i \in I}(x_i - e_i)) > p \cdot (\sum_{i \in I}(x_i - e_i))\}$ and $\bar{A}_{n+1} : \Delta \times X \to 2^{\Delta}$ by $\bar{A}_{n+1}(p, x) := \Delta := X_{n+1}$. For each $i \in I$, define $\bar{P}_i : \Delta \times X \to 2^{X_i}$ by $\bar{P}_i(p, x) := P_i(x)$ and $\bar{A}_i : \Delta \times X \to 2^{X_i}$ by $\bar{A}_i(p, x) := \{x_i \in X_i : p \cdot x_i \leq p \cdot e_i\}$.

Then we have converted the exchange economy $\mathcal{E}$ to an abstract economy $\Gamma = \{(X_i\ , \bar{P}_i, \bar{A}_i) : i = 1, \ldots, n+1\}$ which satisfies all the conditions of the previous theorem.

Thus, there exists $(p^*, x^*) \in \Delta \times X$ such that

(i) $\forall i \in I, x_i^* \in \bar{A}_i(p^*, x^*)$, which is equivalent to (a),

(ii) $\forall i \in I, P_i(x^*) \cap \bar{A}_i(p^*, x^*) = \emptyset$, which is equivalent to (b),

(iii) $\bar{P}_{n+1}(p^*, x^*) \cap \bar{A}_{n+1}(p^*, x^*) = \emptyset$

However, (iii) implies that $\forall q \in \Delta, q \cdot z^* \leq p^* \cdot z^* \leq 0$, where $z^* = \sum_{i \in I}(x_i^* - e_i)$. Now suppose that $z^* \not\leq 0$. Since $-z^* \notin R_+^{\ell}$, by separating hyperplane theorem, there exists $v \in R^{\ell} \setminus \{0\}$ such that $v \cdot (-z^*) < 0$. Without loss of generality, $v \in \Delta$. Thus $v \cdot z^* > 0$, which is a contradiction. Hence $z^* \leq 0$, i.e., $\sum_i x_i^* \leq \sum_i e_i$. □

1.5 Uniqueness of Walrasian Equilibrium

Uniqueness of the equilibrium is obtained under strong assumptions. With less restrictive assumptions we can have economies with multiple equilibria. This may be still satisfactory provided that all the equilibria are locally unique which is equivalent to the finiteness of equilibria when the set of equilibria is compact.

Here, we are going to show that ”almost every” economy has a finite set of equilibria. That is, outside of a null closed subset of the space of economies, every economy has a finite set of equilibria. We will begin with some notation and some preliminary concepts.

Let $F : U \to R^b$ be a continuously differentiable function, where U is an open subset of R^a. A point x is a critical point of F if the Jacobian matrix of F at x has a rank smaller than b, and $y = F(x)$ is a critical value of F. If a point in R^b is not a critical value, then it is called a regular value.

Now, we will be more explicit about the expression "almost every". When we want to measure an interval on a line, the first measure we think of is to take the difference of the end points. What about the measure of more comlicated sets, such as union of intervals, or union of intervals and points etc...

Lebesgue Measure: For each set A of real numbers consider the countable collection $\{I_n\}$ of open intervals that cover A, that is, $A \subset \cup I_n$, and for each such collection consider the sum of the length of the intervals in the collection. Lebesgue measure of A, which is denoted by mA, is defined as the infimum of all such sums.

$$mA = \inf_{a \subset \cup I_n} \sum \ell(I_n)$$

where $\ell(I_n)$ represents the length of interval I_n.

EXAMPLE: There may exist sets with Lebesgue measure zero: e.g. set of rational numbers.

Sard's Theorem: If all the partial derivatives of F to the c^{th} order included, where $c > max(0, a - b)$, exist and are continuous, then the set of critical values of F has Lebesgue measure zero in R^b.

Let L be the set of strictly positive real numbers, P be the set of strictly positive vectors in R^ℓ where ℓ is the number of commodities, and S be the set of vectors in P for which the sum of the components is unity. There are m agents in the economy and agent i's demand function, f_i, is a function from $S \times L$ to $\overline{P}$ such that for every $(p, w_i) \in S \times L$, one has $p \cdot f_i(p, w_i) = w_i$.

Assumption A: If the sequence (p^q, w_i^q) in $S \times L$ converges to (p^0, w_i^0) in $(\overline{S} \setminus S) \times L$, then$|\ f_i(p^q, w_i^q)\ |$ converges to $+\infty$ (Every commodity is desired by agent i).

An economy is defined by $\omega \in P^m$. Given $\omega \in P^m$, an element p of S is an equilibrium price vector of the economy ω if

$$\sum_{i=1}^{m} f_i(p, p \cdot \omega_i) = \sum_{i=1}^{m} \omega_i.$$

We denote by $W(\omega)$ the set of p satisfying this equality. Finally we say that a set A is null if it has Lebesgue measure zero, also we say that a property holds almost everywhere if it holds outside of a null set.

Theorem: Given m continuously differentiable demand functions $(f_1, \ldots, f_m)$, if some f_i satisfies assumption A, then the set of $\omega \in P^m$ for which $W(\omega)$ is infinite has a null closure.

Proof (for details of this proof see Debreu(1970): W.l.o.g. assume that first consumer satisfies assumption A. Let $U = S \times L \times P^{m-1}$, an open set in $R^{\ell m}$. We define the function F from U to $R^{\ell m}$ by $F(e) = (\omega_1, \omega_2, \ldots, \omega_m)$ where $e = (p, w_1, \omega_2, \ldots, \omega_m)$ and

$$\omega_1 = f_1(p, w_1) + \sum_{i=2}^{m} f_i(p, p \cdot \omega_i) - \sum_{i=2}^{m} \omega_i$$

Notice that $\forall e \in U$, $p \cdot \omega_1 = w_1$. Also, given $\omega \in P^m$, the price vector p belongs to $W(\omega)$ if and only if $F(p, p \cdot \omega_1, \omega_2, \ldots, \omega_m) = \omega$ and that the points of $W(\omega)$ are in one-to-one correspondence with the points of $F^{-1}(\omega)$. Since F is continuously differentiable by Sard's theorem, the set C of critical values of F is null.
We now want to prove that $C \cap P^m$ is closed relative to P^m. To this end we establish that if K is a compact subset of P^m, then $F^{-1}(K)$ is compact. This implies that if E contained in U is closed relative to U, then $F(E) \cap P^m$ is closed relative to P^m. Then, as a corollary we have $C \cap P^m$ is closed relative to P^m. If $\omega \in P^m$ is a regular value of F, then $F^{-1}(\omega)$ is finite. If $\omega \in P^m$ is such that $W(\omega)$ is infinite, then $\omega \in C$. Then, $C \cap P^m$ is null and so is its closure.

1.6 Stability of Walrasian Equilibrium

Uniqueness property is obtained under strong assumptions. With less restrictive assumptions we can have economies with multiple equilibria. This may be still satisfactory provided that all the equilibria are locally unique which is equal to the finiteness of equilibria when the set of equilibria is compact. Here, we want to show that almost every economy has a finite set of equilibria.

DEFINITION 1.6.1 : *Stability* means that aggregate excess demand goes down as prices go up.

EXAMPLE 1.6.1 : Suppose $e_1 = (1, 0), e_2 = (0, 1)$ and p be the price of x, q price of y and $u_1(x, y) = min\{x, 2y\}, u_2(x, y) = min\{2x, y\}$. We can derive demand functions as follows. If α units of good y is demanded then 2α units of good x are demanded. For agent 1, a budget line is $2\alpha p + \alpha q = p$. Thus $\alpha = p/(2p + q)$ and $D_1(p, q) = (2p/(2p + q), p/(2p + q))$. Similarly, $D_2(p, q) = (q/(p + 2q), 2q/(p + 2q))$. Then aggregate demand for x is $E_x = 2p/(2p + q) + q/(p + 2q) - 1 = q(p - q)/(2q^2 + 5pq + 2p^2)$. Since denominator is always positive, as p rises, E_x increases so that there is an instability. Similarly, $E_y = p(q - p)/(2p^2 + 5pq + 2q^2)$ so that there is an instability (See figure (a)).

EXAMPLE 1.6.2 : If we change endowment : $e_1 = (0,1), e_2 = (1,0)$, then we get stability. The location of endowment plays role in getting stability (See figure (b)).

THEOREM 1.6.1 (Slutsky Equation) :

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j}\mid_{u=u^0} - x_j \frac{\partial x_i}{\partial m}\mid_{p=p^0}$$

- The LHS is the total effect and is a slope of demand curve.
- The first term of RHS is the substitution effect and is always negative with respect to the own price.
- $\partial x_i/\partial m$ is the income effect, which is positive for normal goods and negative for inferior goods.
- For normal good, we get a negative sloping demand curve.
- For Giffen good, the negative income effect dominates substitution effect, so that we have positive sloping demand and thus the law of demand is violated.
- When the total effect is positive, they are called gross substitutes (chicken and beef). When it is negative, they are gross complements (coffee and sugar).
- Slutsky matrix is symmetric and negative semidefinite and diagonal elements represent own-price effect. Gross substitutability is very important for stability.

DEFINITION 1.6.2 (Hicksian Stability) : The market for good j is *perfectly stable* if all other prices being constant and the following conditions hold :

(1) $E_{jj} = dE_j/dp_j < 0$, where $E_j(p)$ is the excess demand function.

(2) if p_k is flexible for $k \neq j$, p_k adjust in such a way to make $E_k = 0$.

(3) if p_m is flexible for $m \neq j, k$, p_m adjusts so that $E_m = 0$ and so forth.

Note that $dE_j = \sum_k E_{jk} dp_k$ If there are two goods j, k, $dE_k = 0$ so that $dp_j = dE_j E_{kk}/\mid D_2 \mid$. Since $E_{kk} < 0$ by the condition (1), the sufficient condition is $\mid D_2 \mid > 0$.

THEOREM 1.6.3 : If $(-1)^n|D_n| > 0$ where $|D_n|$ is the n-th principal minor of Slutsky matrix, then the economy is Hicksian stable.

N. B. The assumption in the theorem is called *gross substitutability*. Under the assumption of gross substitutability, the equilibrium is unique.

1.7 Optimality of Walrasian Equilibrium

1.7.1 Definitions

DEFINITION 1.7.1 : A $(p, x) \in \Delta \times X$ is a **Walrasian equilibrium** if x is feasible and $x_i' \succ_i x_i \Rightarrow p \cdot x_i' > p \cdot e_i \geq p \cdot x_i, \forall i \in I$.

DEFINITION 1.7.2 : A $(p, x) \in \Delta \times X$ is a **quasi-equilibrium** if x is feasible and $x_i' \succeq_i x_i \Rightarrow p \cdot x_i' \geq p \cdot e_i, \forall i \in I$

DEFINITION 1.7.3 : A (p, x) is a **proper quasi-equilibrium** if it is a quasi-equilibrium and $p \cdot \sum_i x_i \neq p \cdot \sum_i x_i'$ for some feasible allocation x'.

DEFINITION 1.7.4 : A feasible allocation $x \in X$ is said to be **supported by** $p \in R^\ell \setminus \{0\}$ if $x_i' \succeq_i x_i \Rightarrow p \cdot x_i' \geq p \cdot x_i, \forall i \in I$

DEFINITION 1.7.5 : A feasible allocation $x \in X$ is **individually rational** if $x_i \succeq_i e_i, \forall i \in I$.

DEFINITION 1.7.6 : A feasible allocation $x \in X$ is **weakly Pareto optimal** if there is no feasible allocation $x' \in X$ such that $x_i' \succ_i x_i, \forall i \in I$.

DEFINITION 1.7.7 : An feasible allocation $x \in X$ is **Pareto optimal** if there is no feasible allocation $x' \in X$ such that $x_i' \succeq_i x_i, \forall i \in I$ and $x_i' \succ_i x_i$ for some $i \in I$.

THEOREM 1.7.1 : If x is supported by p and $\succeq$ is monotonic, then $p \geq 0$.

THEOREM 1.7.2 : Let (p, x) be a quasi-equilibrium. If $\succeq_i$ is reflexive for every i, then $p \cdot x_i = p \cdot e_i$ for every i. Moreover, if $\succeq_i$ is monotonic, then $p \geq 0$.

EXAMPLE 1.7.1 : When $\succeq_i$ is not strictly monotone and (p, x) is a quasi-equilibrium, it need not be the case that $p \gg 0$.
THEOREM 1.7.3 : If $\succeq_i$ is reflexive and continuous, and $e_i \in intX_i$ for every i, then $Q(\mathcal{E}) \subset W(\mathcal{E})$.

PROOF: Since x is a quasi-equilibrium, it is feasible, i.e., $\sum_{i\in I} x_i = \sum_{i\in I} e_i$. The reflexivity implies that $p \cdot x_i = p \cdot e_i$ for every i. Let $x'_i \succ_i x_i$. Then $p \cdot x'_i \geq p \cdot e_i$. Now suppose $p \cdot x'_i = p \cdot e_i$. Since $e_i \in intX_i$, there is $\hat{x}_i \in X_i$ such that $p \cdot \hat{x}_i < p \cdot e_i$. Consider $x_i^\alpha = \alpha x'_i + (1-\alpha)\hat{x}_i$. By continuity, $x_i^\alpha \succ_i x_i$ for α close to 1. Therefore, $p \cdot x_i^\alpha \geq p \cdot e_i$. But by construction, $p \cdot x_i^\alpha < p \cdot e_i$. This is a contradiction. □

THEOREM 1.7.4 : For all i, there is at most one satiation consumption and his preference is locally nonsatiated at the nonsatiated consumptions. Then $W(\mathcal{E}) \subset Q(\mathcal{E})$.

THEOREM 1.7.5 : $\boldsymbol{P}(\mathcal{E}) \subset \boldsymbol{WP}(\mathcal{E})$.
PROOF: By definition. □

THEOREM 1.7.6 : If $\succeq_i$ is strictly monotonic and continuous for every i, $\boldsymbol{P}(\mathcal{E}) = \boldsymbol{WP}(\mathcal{E})$.
PROOF: Pick $x \in \boldsymbol{WP}(\mathcal{E}) \setminus \boldsymbol{P}(\mathcal{E})$. Then $\exists x'$ such that $x'_i \succeq_i x_i, \forall i \in I$ and $x'_k \succ_k x_k, \exists k \in I$. By the continuity of preferences, $\exists \varepsilon \in R_+^\ell \setminus \{0\}$ such that $X_k \ni x'_k - \varepsilon \succ_k x_k$. Let $x_i^o := x'_i + \varepsilon/(n-1) \in X_i, \forall i \in I \setminus \{k\}$ and $x_k^o := x'_k - \varepsilon$. Then x^o is feasible. However, by the strict monotonicity of preferences, $x_i^o \succ_i x_i, \forall i \in I$, which contradicts the weak Pareto optimality of x. □

1.7.2 Optimality of Equilibria

THEOREM 1.7.7 (First Welfare Theorem I) : Let $\succeq_i$ be reflexive, continuous, and monotonic for every i. If a feasible allocation x is supported by a price $p \in R^\ell \setminus \{0\}$ such that $p \cdot \sum e_i \neq 0$, then x is weakly Pareto optimal.

PROOF: Suppose otherwise. There exists a feasible $x' \in X$ such that $x'_i \succ_i x_i, \forall i \in I$. Since p supports x, $p \cdot x'_i \geq p \cdot x_i, \forall i \in I$. Since $p \cdot \sum_{i\in I} e_i = \sum_{i\in I} p \cdot x'_i = \sum_{i\in I} p \cdot x_i$, we conclude that $p \cdot x'_i = p \cdot x_i, \forall i \in I$. However, by the continuity of preferences, $\exists \alpha \in (0,1)$ such that $(1-\alpha)x'_i \succ_i x_i, \forall i \in I$ and, by the p-supportability of x, $(1-\alpha)p \cdot x'_i \geq p \cdot x_i, \forall i \in I$. Thus $(1-\alpha)p \cdot x'_i \geq p \cdot x'_i$. Since $p \cdot x'_i \geq 0$, $(1-\alpha)p \cdot x'_i = p \cdot x'_i, \forall i \in I$. This implies that $p \cdot x'_i = 0, \forall i \in I$ so that $p \cdot \sum_{i\in I} e_i = 0$, which is a contradiction. □

THEOREM 1.7.8 (First Welfare Theorem II) : $\boldsymbol{W}(\mathcal{E}) \subset \boldsymbol{WP}(\mathcal{E})$.

PROOF: Take a Walrasian equilibrium x. Suppose it is not weakly Pareto optimal. Then there exists a feasible $x' \in X$ such that $x'_i \succ_i x_i, \forall i \in I$. Since x is a Walrasian equilibrium, $p \cdot x'_i > p \cdot e_i, \forall i \in I$. Thus $p \cdot \sum_{i \in I} e_i = \sum_{i \in I} p \cdot x'_i > \sum_{i \in I} p \cdot e_i$, which is a contradiction. □

EXAMPLE 1.7.2 : A Walrasian equilibrium allocation need not be Pareto optimal.

THEOREM 1.7.9 (First Welfare Theorem III) : If $\succeq_i$ is reflexive and strictly convex for every i, $\boldsymbol{W}(\mathcal{E}) \subset \boldsymbol{P}(\mathcal{E})$.

PROOF: Take a Walrasian equilibrium x with respect to p. Suppose it is not Pareto optimal. Then there exists a feasible $x' \in X$ such that $x'_i \succeq_i x_i, \forall i \in I$ and $x'_i \succ_i x_i, \exists i \in I$. Since x is a Walrasian equilibrium, $p \cdot x'_i > p \cdot e_i \geq p \cdot x_i, \exists i \in I$. But $p \cdot \sum_{i \in I} e_i = \sum_{i \in I} p \cdot x'_i = \sum_{i \in I} p \cdot x_i$ Thus $p \cdot x'_k < p \cdot x_k, \exists k \in I$. Moreover, $x'_k \succeq_k x_k, x_k \succeq_k x_k$ and $x'_k \neq x_k$. By the strict convexity of preferences, $x^\alpha_k \succ_k x_k$ so that $p \cdot x^\alpha_k > p \cdot e_k$. By letting $\alpha \to 1$, we have $p \cdot x'_k \geq p \cdot e_k$, a contradiction. □

THEOREM 1.7.10 (First Welfare Theorem IV) : If $\succeq_i$ is transitive and strictly monotonic for every i, $\boldsymbol{W}(\mathcal{E}) \subset \boldsymbol{P}(\mathcal{E})$.

PROOF: Take a Walrasian equilibrium x with respect to p. Suppose it is not Pareto optimal. Then there exists a feasible $x' \in X$ such that $x'_i \succeq_i x_i, \forall i \in I$ and $x'_i \succ_i x_i, \exists i \in I$. Since x is a Walrasian equilibrium, $p \cdot x'_i > p \cdot e_i \geq p \cdot x_i, \exists i \in I$. But $p \cdot \sum_{i \in I} e_i = \sum_{i \in I} p \cdot x'_i = \sum_{i \in I} p \cdot x_i$ Thus $p \cdot x'_k < p \cdot e_k, \exists k \in I$. However, for every $\varepsilon \in R^\ell_+ \setminus \{0\}$, $x'_k + \varepsilon \succ_k x'_k \succeq_k x_k$ so that $p \cdot (x'_k + \varepsilon) \geq p \cdot e_k$. By letting $\varepsilon \to 0$, we have $p \cdot x'_k \geq p \cdot e_k$, a contradiction. □

THEOREM 1.7.11 (Second Welfare Theorem I) : If $\succeq_i$ is complete, transitive and semi-strictly convex for every i and $\succeq_1$ is nonsatiated, a Pareto optimal allocation can be supported by some $p \in R^\ell \setminus \{0\}$.

PROOF: Let $C^o_1(x_1) := \{x'_1 \in X_1 : x'_1 \succ_1 x_1\}$ and $C_i(x_i) := \{x'_i \in X_i : x'_i \succeq_i x_i\}, \forall i \in I \setminus \{1\}$. Then $C(x) := C^o_1(x_1) + \sum_{i=2}^n C_i(x_i)$ is nonempty and convex since $\succeq_1$ is nonsatiated and $\succeq_i$ is complete, transitive, and semi-strictly convex for every i. Note that $e := \sum_i e_i \notin C(x)$ since x is Pareto optimal. By the separating hyperplane theorem, $\exists p \in R^\ell \setminus \{0\}$ such that $\forall z \in C(x), p \cdot z \geq p \cdot e = p \cdot \sum_{i \in I} x_i$. This implies that $z_1 \succ_1 x_1 \Rightarrow p \cdot z_1 \geq p \cdot x_1$ and $z_i \succeq_i x_i \Rightarrow p \cdot z_i \geq p \cdot x_i, \forall i \in I \setminus \{1\}$. Indeed, if $x'_1 \succ_1 x_1$, then $z = x'_1 + \sum_{i \in I \setminus \{1\}} x_i \in C(x)$, from which it follows that $p \cdot x'_1 \geq p \cdot x_1$. On the other hand, for each $i \in I \setminus \{1\}$, suppose that $x_i' \succeq_i x_i$. By the nonsatiation of $\succeq_1$, $\exists \hat{x}_1 \in X_1$ such that $\hat{x}_1 \succ_1 x_1$. By the semi-strict convexity of $\succeq_1$, $x_1^\alpha := \alpha\hat{x}_1 + (1-\alpha)x_1 \succ_1 x_1, \forall \alpha \in (0,1)$. Let $z = x_1^\alpha + x_i' + \sum_{k \in I \setminus \{1,i\}} x_k$. Then $z \in C(x)$, which implies that $p \cdot x_1^\alpha + p \cdot x_i' \geq p \cdot x_1 + p \cdot x_i$. By letting $\alpha \to 0$ we get $p \cdot x_i' \geq p \cdot x_i$.

To complete the proof, we must show that $x_1^o \sim_1 x_1 \Rightarrow p \cdot x_1^o \geq p \cdot x_1$. Suppose that $x_1^o \sim_1 x_1$. By the nonsatiation of $\succeq_1$, $\exists \bar{x}_1 \in X_1$ such that $\bar{x}_1 \succ_1 x_1^o$. By the convexity of $\succeq_1$, $x_1^\lambda := \lambda\bar{x}_1 + (1-\lambda)x_1^o \succ_1 x_1^o, \forall \lambda \in (0,1)$. Since $x_1^o \sim_1 x_1$, $x_1^\lambda \succ_1 x_1$ so that $p \cdot x_1^\lambda \geq p \cdot x_1$ by the previous result. By letting $\lambda \to 0$, we get $p \cdot x_1^o \geq p \cdot x_1$. $\square$

THEOREM 1.7.12 (Second Welfare Theorem II) : Let $\succeq_i$ be complete, transitive, convex, and nonsatiated for every i. Then a weakly Pareto optimal allocation can be supported by some $p \in R^\ell \setminus \{0\}$.

PROOF: Let $C_i(x_i) := \{x_i' \in X_i : x_i' \succ_i x_i\}, \forall i \in I$. Then $C(x) := \sum_{i \in I} C_i(x_i)$ is nonempty and convex since the preferences are complete, transitive, semi-strictly convex and nonsatiated. Note that $e := \sum_i e_i \notin C(x)$ since x is weakly Pareto optimal. By the separating hyperplane theorem, $\exists p \in R^\ell \setminus \{0\}$ such that $\forall z \in C(x), p \cdot z \geq p \cdot e = p \cdot \sum_{i \in I} x_i$. Now, for each $i \in I$, suppose that $x_i' \succeq_i x_i$. By the nonsatiation of $\succeq_k$, $\exists \hat{x}_k \in X_k$ such that $\hat{x}_k \succ_k x_k, \forall k \in I \setminus \{i\}$. By the semi-strict convexity of $\succeq_k$, $x_k^\alpha := \alpha\hat{x}_k + (1-\alpha)x_k \succ_k x_k, \forall k \in I \setminus \{i\}, \forall \alpha \in (0,1)$. Similarly, we get $x_i^\alpha = \alpha\hat{x}_i + (1-\alpha)x_i' \succ_i x_i' \succeq_i x_i$ with $\hat{x}_i \succ_i x_i'$. Then $\sum_i x_i^\alpha \in C(x)$, which implies that $p \cdot \sum_i x_i^\alpha \geq p \cdot \sum_i x_i$. By letting $\alpha \to 0$ we get $p \cdot x_i' \geq p \cdot x_i$. $\square$

THEOREM 1.7.12 : $W(\mathcal{E}) \cap Q(\mathcal{E}) \subset P(\mathcal{E})$.

COROLLARY 1.7.13 : For all i, there is at most one satiation consumption and his preference is locally nonsatiated at the nonsatiated consumptions. Then $W(\mathcal{E}) \subset P(\mathcal{E})$.

1.7.3 Optimality of Quasi-Equilibria

LEMMA 1.7.14 : Suppose that X_i is convex and $\succeq_i$ is continuous. Suppose that (x_i, p) is such that $x_i' \succ x_i$ implies $p \cdot x_i' \geq p \cdot x_i$ and $p \cdot x_i > \inf p \cdot X_i$. Then $x_i' \succ_i x_i$ implies $p \cdot x_i' > p \cdot x_i$.

THEOREM 1.7.15 : Let X_i be convex and $\succeq_i$ be continuous. A quasi-equilibrium allocation x with $p \cdot x_i > \inf p \cdot X_i$ for some i is a weakly Pareto optimal.

EXAMPLE 1.7.3 : A quasi-equilibrium allocation need not be weakly optimal.

EXAMPLE 1.7.4 : When X_i is not convex, $x_i \succ x_i$ but $p \cdot x_i' = p \cdot x_i$.

EXAMPLE 1.7.5 : When $\succeq_i$ is not continuous, $x_i \succ x_i$ but $p \cdot x_i' = p \cdot x_i$.

EXAMPLE 1.7.6 : When $p \cdot x_i \leq \inf p \cdot X_i$, $x_i \succ x_i$ but $p \cdot x_i' = p \cdot x_i$.

THEOREM 1.7.16 : Let X_i be convex and $\succeq_i$ be continuous. A quasi-equilibrium (p, x) with $p \cdot x_i > \inf p \cdot X_i$ for all i is a Walrasian equilibrium.

THEOREM 1.7.17 : If (p, x) is a proper quasi-equilibrium, then for some i, $p \cdot x_i > \inf p \cdot X_i$.

THEOREM 1.7.18 : If $\sum_i X_i \cap intY \neq \emptyset$, then $Q(\mathcal{E}) \subset PQ(\mathcal{E})$.

THEOREM 1.7.19 : Let $\succeq_i$ be strictly monotone and continuous for every i. Let $\sum_i X_i \cap intY \neq \emptyset$. Then $Q(\mathcal{E}) \subset W(\mathcal{E})$.

1.7.4 Equilibrium Properties of Optima

EXAMPLE 1.7.7 : A Pareto optimal allocation need not be a Walrasian equilibrium allocation.

THEOREM 1.7.20 : If there is a consumer whose preference is locally non-satiated, every Pareto optimal allocation is a quasi-equilibrium allocation with respect to some p.

EXAMPLE 1.7.8 : A (weakly) Pareto optimal allocation can only be supported by $p = 0$.

THEOREM 1.7.21 : If $\succeq_i$ is locally nonsatiated for every i, a weakly Pareto optimal allocation is a quasi-equilibrium allocation with respect to some p.

EXAMPLE 1.7.9 : Even if $\succeq_i$ is locally nonsatiated for every i, a Pareto optimal allocation need not be a proper quasi-equilibrium.

THEOREM 1.7.22 : Let $\succeq_i$ be monotonic and proper on $X_i = R^\ell_+$ for every i. Suppose that there are allocations x and x' such that $\sum_i x_i - \sum_i x'_i \in \Gamma$ and $Y = \{e\} - R^\ell_+$, then a weakly Pareto optimal allocation is a proper quasi-equilibrium for some p.

1.7.5 Reviews

THEOREM $W(\mathcal{E}) \subset WP(\mathcal{E})$.

THEOREM $[W(\mathcal{E}) \cap Q(\mathcal{E})] \subset P(\mathcal{E})$.

THEOREM $W(\mathcal{E}) \subset Q(\mathcal{E})$ if one of the following is satisfied :

(1) $\succeq_i$ is strictly convex for every i.

(2) $\succeq_i$ is transitive, semi-strictly convex, and nonsatiated for every i.

(3) $\succeq_i$ is transitive and strictly monotonic for every i.

(4) $\succeq_i$ is transitive and semi-strictly monotonic for every i.

(5) $\succeq_i$ is transitive and locally nonsatiated for every i.

(6) $\succeq_i$ is transitive and has a extremely desirable bundle for every i.

COROLLARY If one of the conditions satisfied, $W(\mathcal{E}) \subset P(\mathcal{E})$.

THEOREM If $\succeq_i$ is reflexive and strictly convex for every i, $W(\mathcal{E}) \subset P(\mathcal{E})$.

THEOREM If $\succeq_i$ is continuous for every i, then an allocation x with $x_i \in intX_i$ is supported by some $p \in R^\ell \setminus \{0\}$.

THEOREM If $\succeq_i$ is reflexive and continuous, and $e_i \in intX_i$ for every i, then $Q(\mathcal{E}) \subset W(\mathcal{E})$.

COROLLARY If $\succeq_i$ is reflexive and continuous, and $e_i \in intX_i$ for every i, then $Q(\mathcal{E}) \subset P(\mathcal{E})$.

THEOREM If $\succeq_i$ is continuous and $e_i \in intX_i$ for some i, then $Q(\mathcal{E}) \subset P(\mathcal{E}) \subset WP(\mathcal{E})$.

THEOREM A Pareto optimal allocation can be supported by some $p \in R^\ell \setminus \{0\}$ if one of the following is satisfied.

(1) $\succeq_i$ is semi-strictly convex for every i and $\succeq_1$ is nonsatiated.

(2) $\succeq_i$ is transitive and convex for every i and $\succeq_1$ is locally nonsatiated.

COROLLARY If $\succeq_i$ is transitive, convex, and continuous for every i, and $\succ_1$ is locally nonsatiated, a Pareto optimal allocation x^* with $x_i^* \in intX_i$ for every i is a Walrasian equilibrium allocation for some $p^* \in R^\ell \setminus \{0\}$ in the economy $\mathcal{E}^* = \{(X_i, \succeq_i, e_i^*) : i \in I\}$ with $e_i^* = x_i^*$ for every i.

THEOREM A weakly Pareto optimal allocation can be supported by some $p \in R^\ell \setminus \{0\}$ if one of the following is satisfied.

(1) $\succeq_i$ is semi-strictly convex and nonsatiated for every i.

(2) $\succeq_i$ is transitive, convex, and locally nonsatiated for every i.

COROLLARY If $\succeq_i$ is transitive, convex, continuous, locally nonsatiated for every i, a weakly Pareto optimal allocation x^* with $x_i^* \in intX_i$ for every i is a Walrasian equilibrium allocation for some $p^* \in R^\ell \setminus \{0\}$ in the economy $\mathcal{E}^* = \{(X_i, \succeq_i, e_i^*) : i \in I\}$ with $e_i^* = x_i^*$ for every i.

2 Core, Value, and Fair Allocations

2.1 Core Allocations

DEFINITION 2.1.1 : An allocation $x \in X$ is a **core allocation** for $\mathcal{E} = \{(X_i, u_i, e_i) : i \in I\}$ if

(1) $\sum_{i \in I} x_i = \sum_{i \in I} e_i$

(2) There does not exists a coalition $S \subset I$ and $(x'_i)_{i \in S} \in \Pi_{i \in S} X_i$ such that $\sum_{i \in S} x'_i = \sum_{i \in S} e_i$ and $x'_i \succ_i x_i, \forall i \in S$.

and denote by $C(\mathcal{E})$ be the set of all core allocations for $\mathcal{E}$.[7]

N.B. We can replace the second conditon with $(2')$ There is no coalition $S \subset I$ and $(x'_i)_{i \in S} \in \Pi_{i \in S} X_i$ such that $\sum_{i \in S} x'_i = \sum_{i \in S} e_i$ and $x'_i \succeq_i x_i, \forall i \in S$ and $x'_i \succ_i x_i, \exists i \in S$. But this condition is less reasonable.

N.B. If an allocation is individually rational and pareto optimal for two agents, then it is the core. The core says that the coalition of a single agent or the grand coalition of two cannot block, and individual rationality says that a singleton coalition cannot improve upon, and grand coalition cannot either. Generally, the set of core allocations is a subset of the set of individually rational and Pareto optimal allocations.

N.B. : Even if one agent has whole endowment of the economy and the others have zero, it is still a Pareto optimum but not fair. The core depends on the initial endowments but the Pareto optimum does not.

THEOREM 2.1.1 : $\boldsymbol{C}(\mathcal{E}) \subset \boldsymbol{WP}(\mathcal{E})$.

[7]
$$C(\mathcal{E}) = V(I) \setminus \bigcup_{S \subset I} intV(S),$$

where

$$\begin{aligned} V(S) &= \{w \in R^{|S|} : \exists (x_i)_{i \in S} \in \Pi_{i \in S} X_i \text{ such that } \sum_{i \in S} x_i = \sum_{i \in S} e_i \text{ and } w_i \leq u_i(x_i), \forall i \in S\}, \\ intV(S) &= \{w \in R^{|S|} : \exists (x_i)_{i \in S} \in \Pi_{i \in S} X_i \text{ such that } \sum_{i \in S} x_i = \sum_{i \in S} e_i \text{ and } w_i < u_i(x_i), \forall i \in S\}. \end{aligned}$$

PROOF: By definition. $\square$

THEOREM 2.1.2 : In a two-person exchange economy, $C(\mathcal{E}) = [WP(\mathcal{E}) \cap IR(\mathcal{E})]$.

PROOF: Let $x \in \boldsymbol{C}(\mathcal{E})$. From the previous theorem, we know that $x \in \boldsymbol{WP}(\mathcal{E})$. Now suppose that $x \notin \boldsymbol{IR}(\mathcal{E})$. Then there is an agent i such that $e_i \succ_i x_i$. Then $\{i\}$ is a coalition who can block the allocation x, which is a contradiction. To prove the reverse direction, choose $x \in WP(\mathcal{E}) \cap IR(\mathcal{E})$ and suppose $x \notin \boldsymbol{C}(\mathcal{E})$. Then there is a coalition $S \subset I$ and $(x'_i)_{i\in S} \in \Pi_{i\in S} X_i$ such that $\sum_{i\in S} x'_i = \sum_{i\in S} e_i$ and $x'_i \succ_i x_i, \forall i \in S$. If $S = \{i\}$, x is not individually rational, a contradiction. If $S = I$, x is not weakly Pareto optimal, a contradiction. $\square$

THEOREM 2.1.3 : $\boldsymbol{W}(\mathcal{E}) \subset \boldsymbol{C}(\mathcal{E})$.

PROOF: Choose $x \in \boldsymbol{W}(\mathcal{E})$ and corresponding prices p. Suppose $x \notin \boldsymbol{C}(\mathcal{E})$. Then there is a coalition $S \subset I$ and $(x'_i)_{i\in S} \in \Pi_{i\in S} X_i$ such that $\sum_{i\in S} x'_i = \sum_{i\in S} e_i$ and $x'_i \succ_i x_i, \forall i \in S$. Then since x is Walrasian equilibrium allocation, $p \cdot x'_i > p \cdot e_i, \forall i \in S$. Thus, $p \cdot \sum_{i\in S} x'_i > p \cdot \sum_{i\in S} e_i$, which is a contradiction. $\square$

N. B. First Welfare Theorem II is a corollary of these two theorems.

THEOREM 2.1.4 : $C(\mathcal{E})$ is nonempty and compact.

THEOREM 2.1.5 : Let $C(\mathcal{E}^r)$ be the core of r-th replica economy. Define $C^r(\mathcal{E}) = \{x \in X : x_i = x^*_{ij}, \forall i, \text{where } x^* \in C(\mathcal{E}^r)\}$. Then $C^{r+1}(\mathcal{E}) \subset C^r(\mathcal{E}), \forall r \in \mathbf{N}$.

DEFINITION 2.1.2 : The set of **Edgeworth equilibria** is the set $E(\mathcal{E}) = \bigcap_{r=1}^{\infty} C^r(\mathcal{E})$.

LEMMA 2.1.6 : If any finite intersection in a family of nonempty compact sets is nonempty, then the intersection of the whole family is nonempty.

THEOREM 2.1.7 : $E(\mathcal{E}) \neq \emptyset$.

THEOREM 2.1.8 (Edgeworth Conjecture) : $E(\mathcal{E}) = W^*(\mathcal{E})$, where $W^*(\mathcal{E})$ is the set of Walrasian equilibrium allocation in $\mathcal{E}$.

2.2 Core Equivalence in a Large Economy

I) FINITE ECONOMY.

An Elementary Core Equivalence Theorem, Anderson, *Econometrica* **1978**
We begin with some notation:
For $x \in R^k$, $\|x\|_\infty = \max_i |x^i|$. $u = (1, \ldots, 1)$. Let $\mathcal{P}$ denote the set of preferences satisfying:
i) weak monotonicity: $x \gg y \Rightarrow x \succ y$.
ii) free disposal: $x \gg y, y \succ z \Rightarrow x \succ z$.
An exchange economy is a map $\varepsilon : A \to \mathcal{P} \times R^k_+$, where A is the set (finite) of agents. An allocation is a map $f : A \to R^k_+$ such that $\sum_{a \in A} f(a) = \sum_{a \in A} e(a)$. A coalition is a non-empty subset of A.
An allocation f is blocked by a coalition S if there exists $g : S \to R^k_+$ with $\sum_{a \in S} f(a) = \sum_{a \in S} e(a)$ such that $g(a) \succ f(a), \forall a \in S$. The core of ε, $C(\varepsilon)$, is the set of all allocations which are not blocked by any coalition. Let $\mathcal{L}$ be the set of all prices, defined by, $\mathcal{L} = \{p \in R^k_+ : \|p\|_1 = 1\}$.

Theorem 1: *Let $\varepsilon : A \to \mathcal{P} \times R^k_+$ be a finite exchange economy, with $|A| = n$. Let $M = \sup\{\|e(a_1) + \ldots + e(a_k)\|_\infty, a_1, \ldots, a_k \in A\}$. If $f \in C(\varepsilon)$, there exists $p \in \mathcal{L}$ such that:*
i) $\frac{1}{n} \sum_{a \in A} |p(f(a) - e(a))| \leq \frac{2M}{n}$
ii) $\frac{1}{n} \sum_{a \in A} |\inf\{p(x - e(a)) : x \succ_a f(a)\}| \leq \frac{2M}{n}$

Proof: Let $f \in C(\varepsilon)$. For $a \in A$, let $\phi(a) = \{x - e(a) : x \succ_a f(a)\} \cup \{0\}$ and define $\Phi = \frac{1}{n} \sum_{a \in A} \phi(a)$.
First, we will show that $\Phi \cap R^k_{--} = \emptyset$.
Suppose not. Then, there exists $G \in \Phi$ such that $G \ll 0$. By the definition of Φ, there exists $g : A \to R^k$ with $g(a) \in \phi(a), \forall a$ and $G = \frac{1}{n} \sum_{a \in A} g(a)$. Let $B = \{a \in A : g(a) \neq 0\}$ and $h(a) = g(a) + e(a) - \frac{n}{|B|} G$ for all $a \in B$. Since $G \ll 0$, $h(a) \gg g(a) + e(a)$ and since $g(a) \in \phi(a)$, $g(a) + e(a) \succ_a f(a)$. We know that $\succ_a \in \mathcal{P}$ and therefore $h(a) \succ_a f(a), \forall a \in B$ (free disposal). Also, the allocation h is feasible among the agents in B, since,

$$\sum_{a \in B} h(a) = \sum_{a \in B} (g(a) + e(a) - \frac{n}{|B|} G)$$

$$= \sum_{a \in B} g(a) + \sum_{a \in B} e(a) - nG = nG + \sum_{a \in B} e(a) - nG = \sum_{a \in B} e(a)$$

Hence, B blocks f, which means that $f \notin C(\varepsilon)$ which is a contradiction. So, $G \ll 0 \Rightarrow G \notin \Phi$.
Let $z = (\frac{M}{n})u$. We need to show that $(con\Phi) \cap \{w \in R^k : w \ll -z\} = \emptyset$.
Suppose not and $x \in (con\Phi) \cap \{w \in R^k : w \ll -z\}$. By the *Shapley-Folkman* Theorem, we can write x in the form $x = \frac{1}{n}\sum_{a\in A} g(a)$, where $g(a) \in con\phi(a), \forall a \in A$ and $\#\{a : g(a) \notin \phi(a)\} = m \leq k$. Let $\{a_1, \ldots, a_m\}$ be those agents that $g(a) \notin \phi(a)$. Define $g' : A \to R^k$ such that $g'(a) = 0$ if $a \in \{a_1, \ldots, a_m\}$ and $g'(a) = g(a)$ otherwise. Since $x \in R^k_+$, $\phi(a_i) \geq -e(a_i)$ and $con\phi(a_i) \geq -e(a_i)$. Let $y = \frac{1}{n}\sum_{a\in A} g'(a) \in \Phi$. Then,

$$y = x - \frac{1}{n}\sum_{i=1}^{m} g(a_i) \leq x + \frac{1}{n}\sum_{i=1}^{m} e(a_i) \leq x + z \ll 0$$

Since $y \in \Phi$, this is contradiction.
$con\Phi$ and $\{w \in R^k : w \ll -z\}$ are convex sets and their intersection is empty. By the *Separating Hyperplane* Theorem, there exists $p \in \mathcal{L}$ such that p separates the two sets. p also separates Φ from $\{w \in R^k : w \ll -z\}$. So, $\inf p\cdot \Phi \geq \sup\{p\cdot w \in R^k : w \ll -z\} = -pz = -(\frac{M}{n})$. Since $0 \in \phi(a), \forall a, 0 \geq \inf p\cdot \Phi \geq -(\frac{M}{n})$.
Notice that $f(a) - e(a) + u/m \in \phi(a)$ for any natural number m. This is because $f(a) + u/m \succ_a f(a)$, for any m, due to the weak monotonicity assumption. Then, $p(f(a) - e(a) + u/m) \geq \inf p\cdot \phi(a)$. By letting $m \to \infty$, we get $p(f(a) - e(a)) \geq \inf p\cdot \phi(a)$. Let $S = \{a \in A : p(f(a) - e(a)) < 0\}$.
Then we have,

$$\frac{1}{n}\sum_{a\in S} p(f(a) - e(a)) \geq \frac{1}{n}\sum_{a\in S} \inf p\cdot \phi(a) \geq \frac{1}{n}\sum_{a\in A} \inf p\cdot \phi(a) \geq -\frac{M}{n}$$

where the second inequality follows since $\inf p\cdot \phi(a)$ is non-positive.
And by feasibility of f we have,

$$\frac{1}{n}\sum_{a\in A} p(f(a) - e(a)) = \frac{1}{n}p(\sum_{a\in A} f(a) - \sum_{a\in A} e(a)) = p\cdot 0 = 0$$

Therefore,

$$\frac{1}{n}\sum_{a\in A} |p(f(a) - e(a))| = \frac{1}{n}[-\sum_{a\in S} p(f(a) - e(a)) + \sum_{a\notin S} p(f(a) - e(a))]$$

$$= -\frac{2}{n}\sum_{a\in S} p(f(a) - e(a)) \leq \frac{2M}{n}$$

where the second equality follows from the feasibility condition.
This proves the first condition of the Theorem.

Let $\Lambda = \{a \in A : \inf p\cdot \phi(a) < 0\}$. If $a \in \Lambda$ then $|\inf\{p(x - e(a)) : x \succ_a f(a)\}| = |\inf p\cdot \phi(a)|$. Therefore,

$$\frac{1}{n}\sum_{a\in A}|\inf\{p(x-e(a)) : x \succ_a f(a)\}| = \frac{1}{n}[-\sum_{a\in\Lambda}\inf p\cdot \phi(a)+\sum_{a\notin\Lambda}|\inf\{p(x-e(a)) : x \succ_a f(a)\}|]$$

since $\Lambda^c \subseteq S^c$ and $\inf p\cdot \phi(a) \leq 0$,

$$\leq \frac{1}{n}[-\sum_{a\in A}\inf p\cdot \phi(a) + \sum_{a\notin S} p(f(a) - e(a))]$$

$$\leq \frac{M}{n} + \frac{M}{n} = \frac{2M}{n}$$

□

By letting the number of agents n go to infinity, we obtain the Core Equivalence. That is the purpose of the next Theorem the proof of which follows directly from Theorem 1.

Theorem 2: *Let $\varepsilon_n : A_n \to \mathcal{P} \times R^k_+$ be a sequence of exchange economies such that $\frac{M_n}{|A_n|} \to 0$. If $f_n \in C(\varepsilon_n)$, there exists prices $p_n \in \mathcal{L}$ such that,*
i) $\frac{1}{|A_n|}\sum_{a\in A_n}|p_n(f_n(a) - e_n(a))| \to 0$
ii) $\frac{1}{|A_n|}\sum_{a\in A_n}|\inf\{p_n(x - e_n(a)) : x \succ_a f_n(a)\}| \to 0$.

II) CONTINUUM ECONOMY. [8]

DEFINITION 2.2.1 : E is the commodity space, which is an ordered Banach space. An economy $\mathcal{E}$ is a quadruple $\{(A, \mathcal{A}, \nu), X, (\succ)_{a\in A}, e\}$ where

(1) $(A, \mathcal{A}, \nu)$ is a **measure space of agents**,

(2) $X : A \to 2^E$ is the **consumption correspondence**,

(3) $\succ_a \subset X(a) \times X(a)$ is the *preference relation* of agent a, and

(4) $e : A \to E$ is the **initial endowment**, where e is Bochner integrable and $e(a) \in X(a)$ for all $a \in A$.

[8] A regular reader may skip this section. However, we can replace E by R^ℓ and the proof remains the same. For details see Rustichini-Yannelis J.M.E., (1991).

DEFINITION 2.2.2 : An **allocation** for the economy $\mathcal{E}$ is a Bochner integrable function $x : A \mapsto E_+$.

DEFINITION 2.2.3 : An allocation x is said to be **feasible** if

$$\int_A x(a)d\nu(a) = \int_A e(a)d\nu(a)$$

DEFINTION 2.2.4 : A **coalition** S is an element of A such that $\nu(S) > 0$

DEFINITION 2.2.5 : The coalition S can **improve upon** the allocation x if there exists an allocation x' such that

(1) $x'(a) \succ_a x(a),\ \nu\text{-}a.e.$ in S

(2) $\int_S x'(a)d\nu(a) = \int_S e(a)d\nu(a)$.

DEFINITION 2.2.6 : The set of all feasible allocations for the economy E that no coalition can improve upon is the **core** of the economy E and it is denoted by $\boldsymbol{C}(\mathcal{E})$.

DEFINITION 2.2.7 : An allocation x and a price $p \in E_+^* \setminus \{0\}$ is a *competitive equilibrium* (**Walrasian equilibrium**) for the economy $\mathcal{E}$, which is denoted by $\boldsymbol{W}(\mathcal{E})$ if

(1) $x(a)$ is a maximal element for $\succ_a$ in the budget set $\{x' \in X(a) : p \cdot x' \leq p \cdot e(a)\},\ \nu\text{-}a.e.$,

(2) $\int_A x(a)d\nu(a) = \int_A e(a)d\nu(a)$.

ASSUMPTIONS :

A.1 E is an ordered Banach space whose positive cone E_+ has a nonempty norm interior, i.e., $intE_+ \neq \emptyset$.

A.2 $(A, \mathcal{A}, \nu)$ is a finite atomless measure space.

A.3 $X(a) = E_+, \forall a \in A$.

A.4 $\int_A ed\nu \gg 0$.

A.5 For each $x \in E_+$, the set $\{x' \succ_a x\}$ is norm open in E_+ for all $a \in A$.

A.6 $\succ_a$ is irreflexive and transitive for all $a \in A$.

A.7 If $x \in E_+$ and $v \in E_+ \setminus \{0\}$, then $x + v \succ_a x$ for all $a \in A$.

THEOREM 2.2.1 (Core Equivalence Theorem) : Under the assumptions **A.1** - **A.7**, $\boldsymbol{C}(\mathcal{E}) = \boldsymbol{W}(\mathcal{E})$.

PROOF: The fact that $W(\mathcal{E}) \subset C(\mathcal{E})$ is well known. Let $x \in C(\mathcal{E})$. To show that for some price p, the pair (p, x) is a competitive equilibrium for $\mathcal{E}$, define the correspondence $\psi : A \to 2^{E_+}$ by

$$\psi(a) := \{x' \in E_+ : x' \succ_a x(a)\} \cup \{e(a)\}.$$

Then

$$\left(\int_A \psi d\nu - \int_A e d\nu\right) \cap int E_- = \emptyset$$

Clearly, $int E_-$ is nonempty and convex. By the definition of ψ, $0 \in (\int_A \psi d\nu - \int_A e d\nu)$. Since $(A, \mathcal{A}, \nu)$ is atomless, by Lemma [2.1 Rustichini and Yannelis (1991)], $c\ell \int_A \psi d\nu$ is convex so that $(\int_A \psi d\nu - \int_A e d\nu)$ is convex. Thus, by the separation hyperplane theorem, there exists a continuous linear functional $p \in E^* \setminus \{0\}, p \geq 0$ such that

$$p \cdot x' \geq p \cdot \int_A e d\nu, \ \forall x' \in \int_A \psi d\nu$$

Now to show that $p \cdot x(a) = p \cdot e(a)$, ν-*a.e.*, let $S \subset A$, $\nu(S) > 0$, $\varepsilon > 0$ and $v \in E_{++}$. Define $x^o : A \mapsto E$ by

$$x^o(a) := \begin{cases} x(a) + \varepsilon v & \text{if } a \in S \\ e(a) & \text{if } a \notin S \end{cases}$$

Then $x^o \in L_1(\psi)$, $\forall S \subset A$. Hence,

$$p \cdot \left(\int_S x d\nu + \varepsilon v \nu(S) + \int_{A \setminus S} e d\nu\right) > p \cdot e$$

Rearranging, we have that $\int_S p \cdot x \geq \int_S p \cdot e$ for any $S \subset A$ since $\varepsilon > 0$ is arbitrary. Thus, it follows that $p \cdot x(a) \geq p \cdot e(a)$, ν-*a.e.* since S is arbitrary. Since x is feasible, $p \cdot x(a) = p \cdot e(a)$, ν-*a.e.*

Consider

$$\hat{x}(a) := \begin{cases} x'(a) & \text{if } a \in S \\ e(a) & \text{if } a \notin S \end{cases}$$

where $x'(a) \succ_a x(a)$ for all $a \in S$. We then have that $\int_A p \cdot x' + \int_{A \setminus S} p \cdot e \geq \int_A p \cdot e$ so that $\int_S p \cdot x' \geq \int_S p \cdot e$, $\forall x' \in L_1(\psi)$. Hence we can conclude that ν-*a.e.*, $p \cdot x' \geq p \cdot e(a), \forall x' \succ_a x(a)$ since S is arbitrary.

To complete the proof, we must show that $x(a)$ is maximal in the budget set $\{x' \in E_+ : p \cdot x' \leq p \cdot e(a)\}$. Since $\int_A e d\nu >> 0$, it follows that $\nu(\{a \in A : p \cdot e(a) > 0\}) > 0$ since $p \in E_+^* \setminus \{0\}$.

Take an agent a with $p \cdot e(a) > 0$. Then $\exists x^o$ such that $p \cdot x^o < p \cdot e(a)$. Suppose that $p \cdot x' \leq p \cdot e(a)$ and let $x^\alpha := \alpha x^o + (1-\alpha)x', \forall \alpha \in (0,1)$. Then $p \cdot x^\alpha < p \cdot e(a), \forall \alpha \in (0,1)$ and $x^\alpha \not\succ_a x(a)$. It follows from the norm continuity of $\succ_a$ that $x' \not\succ_a x(a)$. This proves that that $x(a)$ is maximal in the budget set of a. This, together with the monotonocity of preferences, implies that $p \gg 0$. Indeed, if there exists $v \in E_+ \setminus \{0\}$ such that $p \cdot v = 0$, then $p \cdot (x(a) + v) = p \cdot e(a)$ and by monotonocity $x(a) + v \succ_a x(a)$ contradicting the maximality of $x(a)$ in the budget set.

Thus $p \gg 0$ and $x(a)$ is maximal in the buget set whenever $p \cdot e(a) > 0$. Now consider the agent a with $p \cdot e(a) = 0$. Since $p \gg 0$ and $p \cdot x(a) = p \cdot e(a) = 0$, $x(a) = 0$ is the maximal element in the budget set. Hence, (p, x) is a competitive equilibrium for E. □

LEMMA 2.2.2 :

$$(\int_A \psi d\nu - \int_A e d\nu) \cap int E_- = \emptyset$$

Proof: See [Rustichini and Yannelis, (1991)]. □

2.3 Value Allocations

Consider an economy in which all agents are allowed to cooperate or to bargain with each other. Agent contributes to coalition and can be a member of any coalition. If one sums up all the contributions of an agent to the coalitions that he/she participates, one can find the agent's Shapley value. Shapley value measures the marginal contribution of an agent to the coalitions that he/she participates.

DEFINITION 2.3.1 : A **transferable utility game** $G = (I, V)$ consists of a set of players I and a superadditive characteristic function $V : 2^I \mapsto \mathbf{R}$ such that $V(\emptyset) = 0, V(S \cup T) \geq V(S) + V(T), \forall S, T \subset I, S \cap T = \emptyset$.

DEFINITION 2.3.2 : **The Shapley value** of agent i in the game G is defined as follows:

$$Sh_i(G) := \sum_{S \subset I, S \ni i} \frac{(|S|-1)!(|I|-|S|)!}{|I|!}(V(S) - V(S \setminus \{i\}))$$

N. B. The Shapley value of agent i is an expected marginal contribution of agent i. $(V(S) - V(S \setminus \{i\}))$ is the marginal contribution of agent i to a coalition S.

THEOREM 2.3.1 : $\sum_{i\in I} Sh_i(G) = V(I)$.

THEOREM 2.3.2 : $Sh_i(G) \geq V(\{i\}), \forall i \in I$.

N. B. In general, it is not true that $\forall S \subset I,\ \sum_{i\in S} Sh_i(G) = V(S)$.

DEFINITION 2.3.3 : An allocation $x \in X$ is a **cardinal value allocation** for $\mathcal{E}$ if

(1) $\sum_{i\in I} x_i = \sum_{i\in I} e_i$

(2) $\exists\ (\lambda_i)_{i\in I} \in R_+^{|I|} \setminus \{0\}$ such that $\lambda_i u_i(x_i) = Sh_i(G), \forall i \in I$,
where $Sh_i(G)$ is the Shapley value of agent i derived from the transferable utility game $G = (I, V_{\lambda u})$ with

$$\forall S \subset I,\ V_{\lambda u}(S) := \max\{\sum_{i\in S} \lambda_i u_i(x_i) : \sum_{i\in S} x_i = \sum_{i\in S} e_i\}.$$

N. B. Check if $G = (I, V_{\lambda u})$ is a transferable utility game.

N. B. Note that $V_{\lambda u}(\{i\}) = \lambda_i u_i(e_i)$.

EXAMPLE 2.3.1 : Consider the three agents two goods economy where the utility functions are given as follows.

$$\begin{aligned} u_1(x_1, y_1) = &\quad \min\{x_1, y_1\} \quad e_1 = (1, 0) \\ u_2(x_2, y_2) = &\quad \min\{x_2, y_2\} \quad e_2 = (0, 1) \\ u_3(x_3, y_3) = &\quad (x_3 + y_3)/2 \quad e_3 = (0, 0) \end{aligned}$$

Let $\lambda_i = 1, \forall i = 1, 2, 3$. First calculate the characteristic function $V_{\lambda u}$.

$$\begin{array}{ll} V_{\lambda u}(\{i\}) = 0, & \forall\ i = 1, 2, 3 \\ V_{\lambda u}(\{1, 2\}) = 1, & V_{\lambda u}(\{1, 3\}) = V_{\lambda u}(\{2, 3\}) = 1/2 \\ V_{\lambda u}(\{1, 2, 3\}) = 1 & \end{array}$$

Then the Shapley value are given by

$$\begin{aligned} Sh_1(G) &= 0 + \frac{1}{6}(1 - 0) + \frac{1}{6}(\frac{1}{2} - 0) + \frac{2}{6}(1 - \frac{1}{2}) = \frac{5}{12} \\ Sh_2(G) &= \frac{5}{12}, \quad Sh_3(G) = \frac{2}{12} \end{aligned}$$

Hence the value allocation is:

$$(x_1, y_1) = (x_2, y_2) = (\frac{5}{12}, \frac{5}{12}), \quad (x_3, y_3) = (\frac{2}{12}, \frac{2}{12})$$

N. B. Note that

$$\boldsymbol{W}(\mathcal{E}) = \boldsymbol{C}(\mathcal{E}) = \{((\alpha, \alpha), (1-\alpha, 1-\alpha), (0,0) : \alpha \in [0,1]\}.$$

so that this example shows that there is a value allocation which is neither a Walrasian allocation nor a core allocation.

EXERCISE :

1. Construct a core allocation in three agents exchange economy where the agents are not equally treated.
2. In the example, show that
   (1) the value allocation is $(x_1, y_1) = (x_2, y_2) = (1/2, 1/2)$, $(x_3, y_3) = (0, 0)$ if the third agent's utility function changes into $\min\{x_3, y_3\}$.
   (2) there is no value allocation if λ_i's are different.

N. B. If the agent 3 manipulates his preference from that in (2) to the original one, he becomes better off.

THEOREM 2.3.3 : $\boldsymbol{V}(\mathcal{E}) \subset \boldsymbol{WP}(\mathcal{E})$.

PROOF: Pick a value allocation x. Suppsoe that it is not weakly Pareto optimal. Then there exists a feasible allocation x' such that $u_i(x'_i) > u_i(x_i), \forall i \in I$. Thus $\sum_{i \in I} \lambda_i u_i(x'_i) > \sum_{i \in I} \lambda_i u_i(x_i) = \sum_{i \in I} Sh_i(G) = V_{\lambda u}(I)$. This is a contradiction to the definition of $V_{\lambda u}$. □

THEOREM 2.3.4 : $V(\mathcal{E}) \subset [P(\mathcal{E}) \cap IR(\mathcal{E})]$, where $\lambda \in R^n_{++}$.

PROOF: In a similar way, we can show that $\boldsymbol{V}(\mathcal{E}) \subset \boldsymbol{P}(\mathcal{E})$. Pick a value allocation x. Suppose that it is not individually rational. Then there exists $i \in I$ such that $u_i(e_i) > u_i(x_i)$ so that $\lambda_i u_i(e_i) > \lambda_i u_i(x_i)$. But this is a contradiction since $\lambda_i u_i(x_i) = Sh_i(G) \geq V_{\lambda u}(\{i\}) =$

$\lambda_i u_i(e_i)$. $\square$

COROLLARY 2.3.5 : $V(\mathcal{E}) \subset [WP(\mathcal{E}) \cap IR(\mathcal{E})]$, where $\lambda \in R^n_{++}$.

THEOREM 2.3.6 : In two-agent exchange economy, $\boldsymbol{V}(\mathcal{E}) \subset \boldsymbol{C}(\mathcal{E})$, where $\lambda \in R^n_{++}$.

PROOF: Take a value allocation x. Suppose it is not a core allocation. Then there is a coalition S and $(x'_i)_{i\in S}$ such that $\sum_{i\in S} x'_i = \sum_{i\in S} e_i$ and $u_i(x'_i) > u_i(x_i), \forall i \in S$. First consider the case $S = \{1\}$. Then $x'_1 = e_1$ and $u_1(x'_1) > u_1(x_1)$. Thus $\lambda_1 u_1(e_1) > \lambda_1 u_1(x_1)$, a contradiction. We can do the same for the case where $S = \{2\}$. The only remaining coalition is $S = I$. In this case, we have $\sum_{i\in I} x'_i = \sum_{i\in I} e_i$ and $u_i(x'_i) > u_i(x_i), \forall i \in I$. Thus $\sum_{i\in I} \lambda_i u_i(x'_i) > \sum_{i\in I} \lambda_i u_i(x_i) = \sum_{i\in I} Sh_i(G) = V_{\lambda u}(I)$. This is a contradiction to the definition of $V_{\lambda u}$. $\square$

N. B. This theorem can be viewed as a corollary since we know that the core is equivalent to the set of weakly Pareto optimal and individually rational allocations in a two-agent economy.

THEOREM 2.3.7 : A value allocation is not necessarily a Walrasian equilibrium allocation.

PROOF: In our example, $((5/12, 5/12), (5/12, 5/12), (2/12, 2/12)) \notin \boldsymbol{W}(\mathcal{E})$. $\square$

THEOREM 2.3.8 : A value allocation is not necessarily a core allocation.

PROOF: In our example, $((5/12, 5/12), (5/12, 5/12), (2/12, 2/12)) \notin \boldsymbol{C}(\mathcal{E})$. Concretely, consider the coalition $S = \{1, 2\}$ and $(x'_1, x'_2) = ((1/2, 1/2), (1/2, 1/2))$, which blocks this value allocation. $\square$

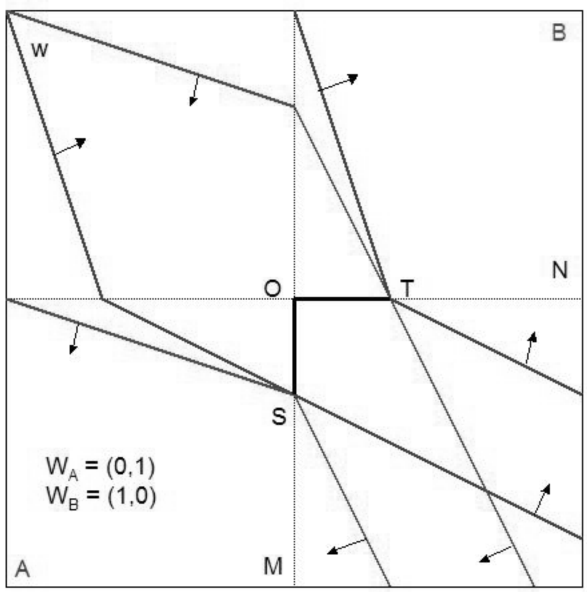
2.4 Manipulability

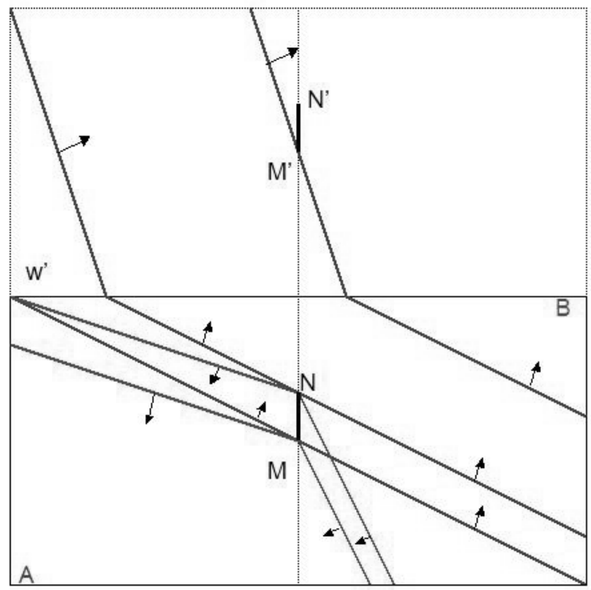
Once we have fixed finite economy, an agent always can manipulate his preference and initial endowment. This equilibrium is manipulated equilibrium. All agent have incentives to misrepresent their characteristic. Therefore, the equilibrium that we found does not represent the true equilibrium. This means that a psycological factor plays an important role in an economy and this has not been addressed by economic theory. Most economic equilibrium concepts stipulate agent's preference maximizing behavior given endowment. Maybe that is not the way of looking at the equilibrium. Thus we will have another concept of equilibrium.

$$u_A(x,y) = \begin{cases} 3x + y & \text{if } y \geq 1/2, \\ 3x + 6y - 5/2 & \text{if } y < 1/2. \end{cases} \qquad w_A = (0,1)$$

$$u_B(x,y) = \begin{cases} x + 3y & \text{if } x > 1/2, \\ 6x + 3y - 5/2 & \text{if } x \leq 1/2. \end{cases} \qquad w_B = (1,0)$$

In this case the Pareto optimal and indiviaually rational allocation (hence, core allocation) is SOT in figure (a). But if agent 1 pretends to have $w'_A = (0, 1/2)$, then the core allocation is MN in figure (b). Since this makes him better off(he'll be on M'N' by adding 1/2 units of y he kept) , he has an incentive to lie about his endowment.





N. B. In the Walrasian equilibrium, an agent can transfer his endowment to the other agents to make the terms of trade favorable to him so that he should be better off. Here we know that λ's play the same role as prices.

N. B. As an economy gets large, agents have diminishing incentive to manipulate endowments.

THEOREM 2.4.1 : Any mechanism is W-manipulable.

2.5 Fair Allocations

Which concept is the best among Walrasian equilibrium, core , value allocation? All they are Pareto optimal. What does "better" mean? Which notion is more equitable? Which way to divide a pie is fair? One criterion is a notion of envy-free.

DEFINITION 2.5.1 : An allocation $x \in X$ is **envy-free** if $u_i(x_i) \geq u_i(x_k),\ \forall i \neq k; i, k \in I$.

DEFINITION 2.5.2 : An allocation $x \in X$ is **fair** if it is Pareto optimal and envy-free.

THEOREM 2.5.1 : A Walrasian equilibrium allocation has equal treatment property.

THEOREM 2.5.2 : A Walrasian equilibrium allocation from the equal division of initial endowments is envy-free.

PROOF: Let (x, p) be a Walrasian equilibrium from the equal division of initial endowments . Suppose x is not envy free. There exists i, k in I such that $u_i(x_k) > u_i(x_i)$. Since x is a Walrasian equilibrium allocation, $p \cdot x_k > p \cdot w$ where $w = \sum_{i \in I} e_i/n$. This is a contradiction since $p \cdot x_k \leq p \cdot w$. $\square$

THEOREM 2.5.3 : A core allocation, with two agents, has the equal treatment property.

COROLLARY 2.5.4 : A core allocation is fair if preferences are strictly monotonic.

THEOREM 2.5.5 : A value allocation does not have the equal treatment property.

THEOREM 2.5.6 : A value allocation from the equal division of initial endowment is not necessarily envy-free.

THEOREM 2.5.7 : A value allocation is not necessarily envy-free.

PROOF: In our example, $u_3(x_1) > u_3(x_3)$. $\square$

N. B. A value allocation gives more consumption bundle to the agents who contribute more to the economy even though the initial endowments are equally distributed. In the value allocation, despite of the fact that two agents have same initial endowment, one agent can be less risk averse than the other so that he can make higher contribution to the society and end up with a higher Shapley value. Therefore, he will be envied by some of the other agents in the economy.

DEFINITION 2.5.3 : An allocation $x \in X$ is **coalitionally fair** if there does not exist disjoint coalitions S_1, S_2 and allocation $(x'_i)_{i \in S_1} \in \Pi_{i \in S_1} X_i$ such that

(1) $\sum_{i \in S_1}(x'_i - e_i) = \sum_{i \in S_2}(x_i - e_i)$,

(2) $u_i(x'_i) > u_i(x_i), \ \forall i \in S_1$.

THEOREM 2.5.8 : $\boldsymbol{W}(\mathcal{E}) \subset \boldsymbol{CF}(\mathcal{E})$.

PROOF: Take a Walrasian equilibrium (x, p). Suppose that x is not coalitionally fair. Then there exist disjoint coalitions S_1, S_2 and allocation $(x'_i)_{i \in S_1} \in \Pi_{i \in S_1} X_i$ such that $\sum_{i \in S_1}(x'_i - e_i) = \sum_{i \in S_2}(x_i - e_i)$ and $u_i(x'_i) > u_i(x_i), \ \forall i \in S_1$. Since x is a Walrasian equilibrium alloction, $p \cdot x'_i > p \cdot e_i, \forall i \in S_1$. Thus $p \cdot \sum_{i \in S_2}(x_i - e_i) = p \cdot \sum_{i \in S_1}(x'_i - e_i) > 0$, which contradicts that $p \cdot \sum_{i \in S_2}(x_i - e_i) \leq 0$. $\square$

THEOREM 2.5.9 : $\boldsymbol{CF}(\mathcal{E}) \subset \boldsymbol{C}(\mathcal{E})$.

PROOF: Set $S_2 = \emptyset$ to have the same condition as in the core. $\square$

COROLLARY 2.5.10 : $CF(\mathcal{E}) \subset [WP(\mathcal{E}) \cap IR(\mathcal{E})]$

THEOREM 2.5.11 : A core allocation is not necessarily coalitionally fair.

THEOREM 2.5.12: A coalitionally fair allocation is not necessarily a Walrasian equilibrium allocation.

THEOREM 2.5.13: A value allocation is not necessarily coalitionally fair.

PROOF: In our example, $((5/12, 5/12), (5/12, 5/12), (2/12, 2/12)) \notin \boldsymbol{CF}(\mathcal{E})$. It is enough to consider the coalitions $S_1 = \{1, 2\}, S_2 = \{3\}$ and $x'_1 = x'_2 = (7/12, 7/12)$. $\square$

2.6 Strong Nash Equilibrium

Definition 2.6.1: A $x^* \in X$ is a **Nash equilibrium** for G if $P_i(x^*) := \{x_i \in X_i : u_i(x_i, x^*_{-i}) > u_i(x^*)\} = \emptyset$ for every $i \in I$.

Definition 2.6.2: A $x^* \in X$ is a **strong (coalitional) Nash equilibrium** for G if there does not exist a coalition S and $(x_i)_{i \in S} \in \Pi_{i \in S} X_i$ such that $u_i((x_i)_{i \in S}, (x^*_i)_{i \in I \setminus S}) > u_i(x^*), \ \forall \, i \in S$.

Definition 2.6.3: A $x^* \in X$ is a α**-core strategy** if there does not exist a coalition S and $(x_i)_{i \in S} \in \Pi_{i \in S} X_i$ such that $u_i((x_i)_{i \in S}, (x'_i)_{i \in I \setminus S}) > u_i(x^*), \ \forall \, i \in S, \ \forall (x'_i)_{i \in I \setminus S} \in \Pi_{i \in \setminus S} X_i$.

Theorem 2.6.1: $\boldsymbol{SNE}(G) \subset \boldsymbol{NE}(G)$.

PROOF: Set $S = \{i\}$ to get the result. $\square$

Theorem 2.6.2: $\boldsymbol{SNE}(G) \subset \boldsymbol{\alpha}\text{-}\boldsymbol{C}(G)$.

PROOF: Choose a strong Nash equilibrium x^*. Suppose it is not an α-core strategy. Then there exist a coalition S and $(x_i)_{i\in S} \in \Pi_{i\in S} X_i$ such that $u_i((x_i)_{i\in S}, (x_i')_{i\in I\setminus S}) > u_i(x^*)$, $\forall\, i \in S$, $\forall (x_i')_{i\in I\setminus S} \in \Pi_{i\in\setminus S} X_i$. In particular, $u_i((x_i)_{i\in S}, (x_i^*)_{i\in I\setminus S}) > u_i(x^*)$, $\forall\, i \in S$, which contradicts that x^* is a strong Nash equilibrium. $\square$

Theorem 2.6.3: A strong Nash equilibrium is efficient.

PROOF: Set $S = I$ to get the result. $\square$

Theorem 2.6.4: An α-core strategy is efficient.

PROOF: Set $S = I$ to get the result. $\square$

N.B. : The previous theorem is a corollary of this theorem.

Theorem 2.6.5: A Nash equilibrium is not necessarily efficient.

Definition 2.6.4: A $x^* \in X$ is a **α-core allocation** if

(1) $\sum_{i\in I} x_i^* = \sum_{i\in I} e_i$,

(2) there does not exist a coalition S and $(x_i)_{i\in S} \in \Pi_{i\in S} X_i$ such that

$$u_i((x_i)_{i\in S}, (x_i')_{i\in I\setminus S}) > u_i(x^*), \forall\, i \in S, \forall (x_i')_{i\in I\setminus S} \in \Pi_{i\in I\setminus S} X_i$$

with $\sum_{i\in I\setminus S} x_i' = \sum_{i\in I\setminus S} e_i$.

3 Core and Value in Differential Information Economies

3.1 Core with Differential Information

When we propose a cooperative solution concept like the core, you have to find an information sharing rule within a coalition. There are three kinds of information sharing rule in a coalition.
Pooling information : This has two problems - incentive to lie, no reward to the superior information.
Private information : noncooperative element in information sharing.
Common knowledge information : difficult to form a blocking coalition.
Let $(\Omega, \mathcal{F}, \mu)$ be an uncertainty environment, $\mathcal{P}(\Omega)$ be the family of finite measurable partitions of Ω, and $\mathcal{M}(\Omega)$ be the set of probability measures on Ω.

Definition 1.1: An **exchange economy with differential inforamtion** $\mathcal{E}$ is given by $\mathcal{E} = \{(X_i, u_i, e_i, \mathcal{F}_i, \mu) : i \in I\}$, where

(1) $X_i := \mathbf{R}^\ell_+$ is the **consumption set** of agent i, $\forall\, i \in I$.

(2) $u_i : \Omega \times \mathbf{R}^\ell_+ \mapsto \mathbf{R}$ is the **random utility function** of agent i, $\forall\, i \in I$.

(3) $e_i : \Omega \mapsto \mathbf{R}^\ell_+$ is $\mathcal{F}_i$-measurable **random endowment fucntion** of agent i, $\forall\, i \in I$.

(4) $\mathcal{F}_i \in \mathcal{P}(\Omega)$ is the **private inforamtion** of agent i, $\forall\, i \in I$.

(5) $\mu \in \mathcal{M}(\Omega)$ is the **common prior** of all agents.

Definition 1.2: The expected utility of agent i for x_i is given by

$$v_i(x_i) := \int_\Omega u_i(\omega, x_i(\omega))d\mu(\omega)$$

Definition 1.3: An **allocation** is a function $x : \Omega \mapsto X$ such that each x_i is $\mathcal{F}$-measurable.
Definition 1.4: An allocation $x : \Omega \mapsto X$ is **feasible** if

$$\sum_{i\in I} x_i(\omega) = \sum_{i\in I} e_i(\omega), \mu\text{-}a.e.$$

Definition 1.5: An allocation $x : \Omega \mapsto X$ is a **coarse core allocation** for the $\mathcal{E}$ if

(1) x_i is $\mathcal{F}_i$-measurable, $\forall\, i \in I$,

(2) $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$, μ-*a.e.*,

(3) there does not exist a coalition S with $(x_i')_{i \in S} : \Omega \mapsto \prod_{i \in S} X_i$ such that

(i) $x_i' - e_i$ is $\wedge_{i \in S} \mathcal{F}_i$-measurable, $\forall\, i \in S$,

(ii) $\sum_{i \in S} x_i'(\omega) = \sum_{i \in S} e_i(\omega)$, μ-*a.e.*,

(iii) $v_i(x_i') > v_i(x_i)$, $\forall\, i \in S$.

Definition 1.6: An allocation $x : \Omega \mapsto X$ is a **private core allocation** for the $\mathcal{E}$ if

(1) x_i is $\mathcal{F}_i$-measurable, $\forall\, i \in I$,

(2) $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$, μ-*a.e.*,

(3) there does not exist a coalition S with $(x_i')_{i \in S} : \Omega \mapsto \prod_{i \in S} X_i$ such that

(i) $x_i' - e_i$ is $\mathcal{F}_i$-measurable, $\forall\, i \in S$,

(ii) $\sum_{i \in S} x_i'(\omega) = \sum_{i \in S} e_i(\omega)$, μ-*a.e.*,

(iii) $v_i(x_i') > v_i(x_i)$, $\forall\, i \in S$.

Definition 1.7: An allocation $x : \Omega \mapsto X$ is a **fine core allocation** for the $\mathcal{E}$ if

(1) x_i is $\mathcal{F}_i$-measurable, $\forall\, i \in I$,

(2) $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$, μ-*a.e.*,

(3) there does not exist a coalition S with $(x_i')_{i \in S} : \Omega \mapsto \prod_{i \in S} X_i$ such that

(i) $x_i' - e_i$ is $\vee_{i \in S} \mathcal{F}_i$-measurable, $\forall\, i \in S$,

(ii) $\sum_{i \in S} x_i'(\omega) = \sum_{i \in S} e_i(\omega)$, μ-*a.e.*,

(iii) $v_i(x_i') > v_i(x_i)$, $\forall\, i \in S$.

Definition 1.8: An allocation $x : \Omega \mapsto X$ is a **strong coarse core allocation** for the $\mathcal{E}$ if

(1) x_i is $\wedge_{i \in I} \mathcal{F}_i$-measurable, $\forall\, i \in I$,

(2) $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$, μ-*a.e.*,

(3) there does not exist a coalition S with $(x_i')_{i \in S} : \Omega \mapsto \prod_{i \in S} X_i$ such that

(i) $x_i' - e_i$ is $\wedge_{i \in S} \mathcal{F}_i$-measurable, $\forall\, i \in S$,

(ii) $\sum_{i \in S} x_i'(\omega) = \sum_{i \in S} e_i(\omega),\ \mu\text{-}a.e.$,

(iii) $v_i(x_i') > v_i(x_i),\ \forall\, i \in S$.

Definition 1.9: An allocation $x : \Omega \mapsto X$ is a **weak fine core allocation** for the $\mathcal{E}$ if

(1) x_i is $\vee_{i \in I} \mathcal{F}_i$-measurable, $\forall\, i \in I$,

(2) $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega),\ \mu\text{-}a.e.$,

(3) there does not exist a coalition S with $(x_i')_{i \in S} : \Omega \mapsto \prod_{i \in S} X_i$ such that

- (i) $x_i' - e_i$ is $\vee_{i \in S} \mathcal{F}_i$-measurable, $\forall\, i \in S$,
- (ii) $\sum_{i \in S} x_i'(\omega) = \sum_{i \in S} e_i(\omega),\ \mu\text{-}a.e.$,
- (iii) $v_i(x_i') > v_i(x_i),\ \forall\, i \in S$.

N.B. : In the coarse core, when they form the coalition they do not allow a trade which they cannot verify. I will not allow you to use any information over and above what I know. so the blocking is much harder and the core is much bigger than those in the fine core. Although a pooling of information seems to be reasonable for the cooperative solution concept, it does not make much sense in a concrete example, since (i) an agent with superior information never be rewarded (free rider problem) and (ii) it is possible for an agent to lie about his information and to become better off at the expense of other people.

***Theorem 1.1:** Let $\mathcal{E} = \{(X_i, u_i, e_i, \mathcal{F}_i, \mu) : i \in I\}$ be an exchange economy with differential information, satisfying the following assumptions for each $i \in I$

A.1 $X_i : \Omega \mapsto 2^{\mathbf{R}_+^\ell}$ is nonempty closed convex valued.

A.2 $u_i : \Omega \times \mathbf{R}_+^\ell \mapsto \mathbf{R}$ is integrably bounded, and continuous and concave in $\mathbf{R}_+^\ell$.

Then a private core allocation exists in $\mathcal{E}$.

Theorem 1.2: $\boldsymbol{FC}(\mathcal{E}) \subset \boldsymbol{PC}(\mathcal{E})$

Theorem 1.3: $\boldsymbol{PC}(\mathcal{E}) \subset \boldsymbol{CC}(\mathcal{E})$

PROOF: Choose a private core allocation x. Suppose it is not a coarse core allocation. There is a coalition S and $(x_i')_{i \in S} : \Omega \mapsto \Pi_{i \in S} X_i$ such that x_i' is $\wedge_{i \in S} \mathcal{F}_i$-measurable, $\forall\, i \in S$, $\sum_{i \in S} x_i'(\omega) = \sum_{i \in S} e_i(\omega)$, μ-$a.e.$ and $v_i(x_i') > v_i(x_i)$, $\forall\, i \in S$. However, since x_i is $\wedge_{i \in S} \mathcal{F}_i$-measurable for every $i \in S$, it is also $\mathcal{F}_i$-measurable for every $i \in S$. Hence, the coalition S and $(x'_i)_{i\in S}$ must be a blocking coalition against x in the private core mechanism, which is a contradiction. □

Theorem 1.4: $\boldsymbol{SCC}(\mathcal{E}) \subset \boldsymbol{CC}(\mathcal{E})$

Theorem 1.5: $\boldsymbol{FC}(\mathcal{E}) \subset \boldsymbol{WFC}(\mathcal{E})$

Corollary 1.6: Under **A.1 - A.2**, a coarse core allocation exists in $\mathcal{E}$.

Corollary 1.7: Under **A.1 - A.2**, a weak fine core allocation exists in $\mathcal{E}$.

Theorem 1.8: $\boldsymbol{FC}(\mathcal{E})$ may be empty.

Theorem 1.9: $\boldsymbol{SCC}(\mathcal{E})$ may be empty.

Example 1.1 : Consider an economy with three agents and three states of nature. There is only one good in each state. All agents have the same utility function $u_i(x) = \sqrt{x}$ and each state occurs with the same probability. The random initial endowments and the private informations of the agents are given as follows.

$$\begin{aligned}
&e_1 = (10, 10, 0), \quad \mathcal{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\} \\
&e_2 = (10, 0, 10), \quad \mathcal{F}_2 = \{\{\omega_1, \omega_3\}, \{\omega_2\}\} \\
&e_3 = (0,\ 0,\ 0), \quad\ \ \mathcal{F}_3 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\} (\text{or } = \{\{\omega_1\}, \{\omega_2, \omega_3\}\})
\end{aligned}$$

First note that any allocation which satisfies the conditions (1), (2) of private core, coarse core and fine core will take the following form with $\varepsilon, \delta \in [0, 10]$ and $\alpha, \beta \in [0, 1]$.

$$\begin{array}{rcccccc}
x_1 & = & (& 10-\varepsilon, & 10-\varepsilon, & \alpha\delta &), \\
x_2 & = & (& 10-\delta, & \beta\varepsilon, & 10-\delta &), \\
x_3 & = & (& \varepsilon+\delta, & (1-\beta)\varepsilon, & (1-\alpha)\delta &)
\end{array}$$

1. Coarse Core

We have to eliminate the blockable allocations from the set of the allocations taking the above form. To survive single-agent coalitions, the allocation must be individually rational. If $S = \{1, 2\}$, there is no possible allocation $(x_i)_{i\in S}$ since $\wedge_{i\in S}\mathcal{F}_i = \{\Omega\}$. When $S = \{1, 3\}$, they cannot improve upon the initial endowment since the agent 3 has nothing to give to the agent 1. The same argument is applied to the coalition $S = \{2, 3\}$. For the grand coalition, there is no possible allocation $(x_i)_{i\in I}$ since $\wedge_{i\in I}\mathcal{F}_i = \{\Omega\}$. Hence the coarse core is

$$CC(\mathcal{E}) = \bigcup_{\alpha,\beta\in[0,1]} CC(\mathcal{E};\alpha,\beta)$$

where

$$\begin{aligned} CC(\mathcal{E};\alpha,\beta) = \;& \{((10-\varepsilon, 10-\varepsilon, \alpha\delta), (10-\delta, \beta\varepsilon, 10-\delta), (\varepsilon+\delta, (1-\beta)\varepsilon, (1-\alpha)\delta)) : \\ & \varepsilon,\delta\in[0,10],\ 2\sqrt{10-\varepsilon}+\sqrt{\alpha\delta}\geq 2\sqrt{10},\ 2\sqrt{10-\delta}+\sqrt{\beta\varepsilon}\geq 2\sqrt{10}\} \end{aligned}$$

2. Private Core

Since we know that any private core allocation is a coarse core allocation, We have only to eliminate the blockable allocations from the coarse core. First the individual rationality of the coarse core implies that we don't have to check the singleton coalitions. Consider the coalition $S = \{1,3\}$. It cannot improve upon (e_1, e_3) simply because the agent 3 has nothing to give to the agent 1 in any state. A similar argument applies to the coalition $S = \{2,3\}$. Thus these coalitions cannot block any coarse core allocation. Now take the coalition $S = \{1,2\}$ and consider the corresponding coalition allocations in the coarse core to the solution (ε,δ) of the following maximization problem, for each $\lambda\in[0,1]$,

$$\max_{\varepsilon,\delta\in[0,10]} \frac{\lambda}{3}(2\sqrt{10-\varepsilon}+\sqrt{\alpha\delta}) + \frac{1-\lambda}{3}(2\sqrt{10-\delta}+\sqrt{\beta\varepsilon}).$$

The resulting allocations for this coalition are efficient for this coalition in some sense and they are satisfying $(1-\varepsilon)(1-\delta) = 16\varepsilon\delta$. First consider the allocations satisfying $(1-\varepsilon)(1-\delta) > 16\varepsilon\delta$. These allocations are blocked by the grand coalition simply because they can increase their utilities by raising ε and δ appropriately until they reach the allocations satistying $(1-\varepsilon)(1-\delta) = 16\varepsilon\delta$. Now take the allocations satisfying $(1-\varepsilon)(1-\delta) = 16\varepsilon\delta$, each of which is an efficient allocation for the coalition $S = \{1,2\}$ in a sense that it survives the coalition $S = \{1,2\}$ and the grand coalition. Finally, consider the allocations satisfying $(1-\varepsilon)(1-\delta) < 16\varepsilon\delta$. The coalition $S = \{1,2\}$ cannot form a blocking coalition by the feasibility condition. For the grand coalition to be a blocking coalition, it is necessary that the utilities of the agent 1 and 2 be improved. But this implies that ε and δ should be smaller, whence the utility of the agent 3 cannot be improved. Therefore, the grand coalition cannot be a blocking coalition. Clearly, these allocations are viable against the other allocations. Hence, the private core is

$$PC(\mathcal{E}) = \bigcup_{\alpha,\beta\in[0,1]} \{x\in CC(\mathcal{E};\alpha,\beta) : (1-\varepsilon)(1-\delta)\leq 16\varepsilon\delta\}$$

In particluar, $x^* = ((8,8,2),(8,2,8),(4,0,0))$ is a private core allocation. Since it is individually rational, no singleton coalition can block this allocation. Consider the coaliton $S = \{1,2\}$. The best allocation they can acheive is (e_1, e_2). Otherwise, by the $\mathcal{F}_i$-measurability, we have $(10-\varepsilon)+(10-\delta) < 20$, which contradicts the feasibility condition. Now take the coalition $S = \{1,3\}$. It cannot improve upon (e_1, e_3) simply because the agent 3 has nothing to give to the agent 1 in any state. A similar argument applies to the coalition $S = \{2,3\}$. Thus these coalitions cannot block this allocation. Finally consider the grand coalition. If it is to be a blocking coalition, then there exists

$$x' = ((10-\varepsilon, 10-\varepsilon, \alpha\delta), (10-\delta, \beta\varepsilon, 10-\delta), (\varepsilon+\delta, (1-\beta)\varepsilon, (1-\alpha)\delta))$$

for some $\alpha, \beta \in [0,1]$ and $\varepsilon, \delta \in [0,10]$ such that

$$\begin{aligned}
\frac{1}{3}(\sqrt{10-\varepsilon} + \sqrt{10-\varepsilon} + \sqrt{\alpha\delta}) &> \frac{1}{3}(2\sqrt{8} + \sqrt{2}) \\
\frac{1}{3}(\sqrt{10-\delta} + \sqrt{\beta\varepsilon} + \sqrt{10-\delta}) &> \frac{1}{3}(2\sqrt{8} + \sqrt{2}) \\
\frac{1}{3}(\sqrt{\varepsilon+\delta} + \sqrt{(1-\beta)\varepsilon} + \sqrt{(1-\alpha)\delta}) &> \frac{1}{3}\sqrt{4}
\end{aligned}$$

Thus, we get

$$\frac{1}{6}(2\sqrt{10-\varepsilon} + \sqrt{\delta}) + \frac{1}{6}(2\sqrt{10-\delta} + \sqrt{\varepsilon}) > \frac{1}{3}(2\sqrt{8} + \sqrt{2})$$

But (x_1^*, x_2^*) is the solution of the maximization where $\lambda = 1/2$ and $\alpha = \beta = 1$ and the maximum is $(2\sqrt{8} + \sqrt{2})/3$. This contradiction establishes that x^* is a private core allocation.

3. Fine Core

We know that any fine core allocation is a private core allocation. Let us choose any private core allocation x. Now consider a blocking coalition $S = \{1,2\}$ and its allocation $(x_i')_{i \in S}$ such that

$$\begin{aligned}
x_1' &= (10, x_1(\omega_2), x_1(\omega_3)), \\
x_2' &= (10, x_2(\omega_2), x_2(\omega_3))
\end{aligned}$$

Here we know that this coalition blocks x since $v_i(x_i') > v_i(x_i), \forall i = 1,2$ since $x_3(\omega_1) > 0$. Hence, we conclude that the fine core is empty.

N.B. : Alternatively, we can start with any allocation x. Suppose $x_3(\omega_1) = 0$. Then the possible allocation is e by the $\mathcal{F}_i$-measurability. But e is blocked by the coalition $S = \{1,2\}$ and the allocation $((10,5,5),(10,5,5))$. If $x_3(\omega_1) > 0$, the above step will do.

4. Strong Coarse Core

Since $\wedge_{i\in I}\mathcal{F}_i = \{\Omega\}$, there is no feasible allocation x where $x_i, \forall i \in I$ is $\wedge_{i\in I}\mathcal{F}_i$-measurable in this initial endowment structure. Hence the strong coarse core is empty.

5. Weak Fine Core

If agent 1 and agent 2 pool their information to have complete information, there is no incetive to give their endowments to get additional information. To survive singleton coalitions, any weak fine allocation is individually rational. Thus

$$\begin{aligned} W = \ & \{(10, 10-\varepsilon, \delta), (10, \varepsilon, 10-\delta), (0,0,0) : \varepsilon, \delta \in [0,10], \\ & \sqrt{10} + \sqrt{10-\varepsilon} + \sqrt{\delta} \geq 2\sqrt{10}, \sqrt{10} + \sqrt{\varepsilon} + \sqrt{10-\delta} \geq 2\sqrt{10}\} \end{aligned}$$

is a candidate set for the weak fine core. However, if we consider the coalition $S = \{1,2\}$, we can easily check that any allocation in W which is not the allocation generated by the following maximization problem, for each $\lambda \in [0,1]$,

$$\max_{\varepsilon,\delta\in[0,10]} \frac{\lambda}{3}(\sqrt{10} + \sqrt{10-\varepsilon} + \sqrt{\delta}) + \frac{1-\lambda}{3}(\sqrt{10} + \sqrt{\varepsilon} + \sqrt{10-\delta}).$$

is can be blocked by this coalition for which the maximization generates the efficient allocation. Hence the weak fine core is

$$\begin{aligned} WFC(\mathcal{E}) = \ & \{(10, 10-\varepsilon, \delta), (10, \varepsilon, 10-\delta), (0,0,0) : \varepsilon, \delta \in [0,10], \varepsilon + \delta = 10, \\ & \sqrt{10} + \sqrt{10-\varepsilon} + \sqrt{\delta} \geq 2\sqrt{10}, \sqrt{10} + \sqrt{10-\delta} + \sqrt{\varepsilon} \geq 2\sqrt{10}\} \end{aligned}$$

In particular, $x^o = ((10,5,5), (10,5,5), (0,0,0))$ is a weak fine core allocation. Indeed, since it is individually rational, it is viable against singleton coalitions. Consider the coalition $S = \{1,2\}$. It is an allocation generated by the maximization so that there is no way to improve both utilities. Consider the coalition $S = \{1,3\}$. It cannot improve upon (e_1, e_3) simply because the agent 3 has nothing to give to the agent 1 in any state. Thus this coalition cannot block this allocation. In a similar way, $S = \{2,3\}$ cannot block this allocation. Finally, the grand coalition cannot block this allocation. Otherwise, there exists a feasible allocation x' such that $v_i(x'_i) > v_i(x^o_i), \forall i \in I$. Then

$$\frac{1}{2}v_1(x'_1 + \frac{x'_3}{2}) + \frac{1}{2}v_2(x'_2 + \frac{x'_3}{2}) > \frac{1}{2}v_1(x^o_1) + \frac{1}{2}v_2(x^o_2)$$

and $(x'_1 + x'_3/2, x'_2 + x'_3/2)$ is feasible in the coalition $S = \{1,2\}$. But (x^o_1, x^o_2) is the solution of the maximization where $\lambda = 1/2$. This contradiction establishes that x^o is a weak fine core allocation.

Example 2. : Consider the same economy except that $\mathcal{F}_3 = \{\{\omega_1, \omega_2, \omega_3\}\}$ First note that the unique allocation which satisfies the conditions (1), (2) of private core, coarse core and fine core is $e = (e_1, e_2, e_3)$.

1. Coarse Core

If $|S| \geq 2$, since $\wedge_{i \in S}\mathcal{F}_i = \{\Omega\}$, there is no possible blocking coalition that can derive $(x_i)_{i \in S}$ such that it is feasible in S and x_i is $\wedge_{i \in S}\mathcal{F}_i$-measurable for every $i \in S$ from the given initial endowment structure. Any single-agent coalition cannot block e since its possible allocation is his endowment itself. Hence, the unique coarse core allocation is e.

2. Private Core

Since we know that a private core allocation is a coarse core allocation. e is the unique candidate for private core allocation. It can be easily checked that e is viable against all the possible blocking coalitions. Hence e is the unique private core allocation.

3. Fine Core

Since we know that any fine core allocation is a private core allocation, e is the unique candidate for fine core allocation. However, consider a blocking coalition $S = \{1, 2\}$ and its allocation $(x'_i)_{i \in S}$ such that

$$\begin{aligned} x'_1 &= (10, 5, 5), \\ x'_2 &= (10, 5, 5). \end{aligned}$$

Here we know that this coalition blocks e since $v_i(x'_i) > v_i(e_i), \forall i = 1, 2$. Hence, we conclude that the fine core is empty.

4. Strong Coarse Core

Since $\wedge_{i \in I}\mathcal{F}_i = \{\Omega\}$, there is no feasible allocation x where $x_i, \forall i \in I$ is $\wedge_{i \in I}\mathcal{F}_i$-measurable in this initial endowment structure. Hence the strong coarse is empty.

5. Weak Fine Core: See the weak fine core in Example 1.

Example 3. :

$$\begin{array}{ll} u_i(x_i) = \log x_i, \forall i \in I, & \mu(\omega_k) = 1/4,\ \forall\, k = 1, 2, 3, 4., \\ e_1 = (20, 20, 2, 20), & \mathcal{F}_1 = \{\{\omega_1, \omega_2, \omega_4\}\{\omega_3\}\}, \\ e_2 = (10, 4, 10, 10), & \mathcal{F}_2 = \{\{\omega_1, \omega_3, \omega_4\}\{\omega_2\}\}, \\ e_3 = (0, 0, 0, 0), & \mathcal{F}_3 = \{\{\omega_1, \omega_4\}\{\omega_2, \omega_3\}\}. \end{array}$$

1. Coarse Core

The coarse core is the set of individually rational allocations with following form.

$$\begin{array}{lllllll} x_1 = & (& 20 - \varepsilon, & 20 - \varepsilon, & 2 + \delta, & 20 - \varepsilon &) \\ x_2 = & (& 10 - \delta, & 4 + \varepsilon, & 10 - \delta, & 10 - \delta &) \\ x_3 = & (& 0 + \varepsilon + \delta, & 0, & 0, & 0 + \varepsilon + \delta &) \end{array}$$

2. Private Core

We can find this private allocation by letting $\lambda = 1/2$ and solving the maximization problem.

$$x = ((18, 18, 3, 18), (9, 6, 9, 9), (3, 0, 0, 3))$$

3. Fine Core

We can show that the fine core is empty by the same argument as in Example 1.

4. Strong Coarse Core

Since $\wedge_{i\in I}\mathcal{F}_i = \{\Omega\}$, there is no feasible allocation x where each x_i $\wedge_{i\in I}\mathcal{F}_i = \{\Omega\}$-measurable in this endowment struture. Hence the strong coarse core is empty.

5. Weak Fine Core

We can verify that $((20, 12, 6, 20), (10, 12, 6, 10), (0, 0, 0, 0))$ is a weak fine core allocation.

3.2 Incentive Compatibility of the Cores.

Definition : A feasible allocation $x : \Omega \mapsto X$ is **incentive compatible** for $\mathcal{E}$ if there does not exist $i \in I$ and states ω, ω' with $\omega' \in E_k(\omega),\ \forall\, k \in I \setminus \{i\}$ such that $u_i(e_i(\omega) + x_i(\omega') - e_i(\omega')) > u_i(x_i(\omega))$.

Definition : A feasible allocation $x : \Omega \mapsto X$ is **coalitionally incentive compatible** for $\mathcal{E}$ if there does not exist a coalition S and states ω, ω' with $\omega' \in E_k(\omega),\ \forall\, k \in I \setminus S$ such that $u_i(e_i(\omega) + x_i(\omega') - e_i(\omega')) > u_i(x_i(\omega)),\ \forall\, i \in S$.

Definition : A feasible allocation $x : \Omega \mapsto X$ is **weak coalitionally incentive compatible** for $\mathcal{E}$ if there does not exist a coalition S and states ω, ω' such that

(1) $\omega' \in E_k(\omega),\ \forall\, k \in I \setminus S$.

(2) $E_i(\omega') \in \wedge_{i\in S}\mathcal{F}_i$ and $\mu(E_i(\omega')) > 0,\ \forall\, i \in S$.

(3) $u_i(e_i(\omega) + x_i(\omega') - e_i(\omega')) > u_i(x_i(\omega)),\ \forall\, i \in S$.

Theorem : $CC(\mathcal{E}) \subset CIC(\mathcal{E})$.

Theorem : $FC(\mathcal{E}) \subset CIC(\mathcal{E})$.

N.B. : This theorem is a corollary.

Theorem : Let $x : \Omega \mapsto \prod_{i \in I} X_i$ be a feasible allocation where $X_i = \mathbf{R}_+$ such that

(1) x_i is $\mathcal{F}_i$-measurable for every $i \in I$,

(2) u_i is monotonic, i.e., $x_i' > x_i$ implies that $u_i(x_i') > u_i(x_i)$ for every $i \in I$.

Then x is coalitionally incentive compatible.
PROOF : Let x be a feasible allocation such that x_i is $\mathcal{F}_i$-measurable for every $i \in I$. Suppose that x is not coalitionally incetive compatible. Then there exists a coalition S and states ω, ω' with $\omega' \in E_k(\omega),\ \forall\, k \in I \setminus S$ such that $u_i(e_i(\omega) + x_i(\omega') - e_i(\omega')) > u_i(x_i(\omega)),\ \forall\, i \in S$. Now we have the following by the feasibility of x and $\mathcal{F}_i$-measurability of x_i.

$$\begin{aligned}
\sum_{i \in S}[x_i(\omega) - e_i(\omega)] &= -\sum_{i \notin S}[x_i(\omega) - e_i(\omega)], \text{ by the feasibility} \\
&= -\sum_{i \notin S}[x_i(\omega') - e_i(\omega')], \text{ by the measurability} \\
&= \sum_{i \in S}[x_i(\omega') - e_i(\omega')], \text{ by the feasibility.}
\end{aligned}$$

Now suppose that $x_i(\omega) - e_i(\omega) < x_i(\omega') - e_i(\omega')$ for some $i \in S$. Then it follows from the previous argument that $x_j(\omega) - e_j(\omega) > x_j(\omega') - e_j(\omega')$ for some $j \in S$, which implies by the monotonocity of u_j that $u_j(x_j(\omega)) > u_j(e_j(\omega) + x_j(\omega') - e_j(\omega))$ for some $j \in S$, a contradiction. If $x_i(\omega) - e_i(\omega) \geq x_i(\omega') - e_i(\omega')$ for some $i \in S$, then $u_i(x_i(\omega)) \geq u_i(e_i(\omega) + x_i(\omega') - e_i(\omega))$ for some $i \in S$ by the monotonicity of u_i, which is a contradiction. Hence, x is coalitionally incentive compatible. $\square$

Theorem : A weak fine core allocation is not necessarily incentive compatible.
PROOF : In Example 1, consider a weak fine core allocation $(10, 5, 5), (10, 5, 5)$ and suppose that the true state is ω_1. Then there is an incentive for the agent 1 to lie to the agent 3 that the ω_3 occurs, where he can get the utility $u_1(10 + 5)$ in ω_1 instead of $u_1(10)$. $\square$

N.B. : This theorem (or proof) is not consistent with the definition of incentive compatibility. Here, note that $\omega_3 \notin E_3(\omega_1)$. An appropriate definition may be : A feasible allocation $x : \Omega \mapsto X$ is **coalitionally incentive compatible** for $\mathcal{E}$ if there does not exist a coalition S and states ω, ω' with $\omega' \in E_k(\omega),\ \forall\, k \in \{i \in I \setminus S : u_i(x_i(\omega)) > u_i(e_i(\omega)) \text{ or } u_i(x(\omega')) < u_i(e_i(\omega'))\}$ such that $u_i(e_i(\omega) + x_i(\omega') - e_i(\omega')) > u_i(x_i(\omega)),\ \forall\, i \in S$.

***Theorem :** Let $\mathcal{E} = \{(X_i, u_i, e_i, \mathcal{F}_i, \mu) : i \in I\}$ be an exchange economy with differential inforamtion satisfying **A.1** and **A.2** for each $i \in I$. Moreover, suppose that preferences are monotone. Then any private core allocation for $\mathcal{E}$ is weak coalitionally incentive compatible.

***Theorem :** Under same conditions, any private Pareto optimal allocation is weakly coalitionally incentive compatible.

3.3 Nash Equilibrium and α-Core with Differential Information

Definition : $x^* \in L_X := \Pi_{i\in I} L_{X_i}$ is a **Bayesian Nash equlibrium** for G if

$$v_i(x^*) \geq v_i(x_i, x^*_{-i}), \forall x_i \in L_{X_i}$$

where $L_{X_i} := \{x_i : \Omega \mapsto \mathbf{R}^\ell_+ : x_i(\omega) \in X_i(\omega), \forall \omega \in \Omega \text{ and } x_i \text{ is } \mathcal{F}_i\text{-measurable.}\}$.
Definition : $x^* \in L_X$ is a **private strong Nash equilibrium** for the G if there does not exist a coalition S and $(x_i)_{i\in S} \in \Pi_{i\in S} L_{X_i}$ such that

$$\forall i \in I,\ v_i((x_i)_{i\in S}, (x^*_i)_{i\in I\setminus S}) > v_i(x^*).$$

Definition : $x^* \in L_X$ is a **private α-core strategy** for the G if there does not exist a coalition S and $(x_i)_{i\in S} \in \Pi_{i\in S} L_{X_i}$ such that

$$\forall i \in I,\ v_i((x_i)_{i\in S}, (x'_i)_{i\in I\setminus S}) > v_i(x^*),\ \forall\, (x'_i)_{i\in I\setminus S} \in \Pi_{i\in I\setminus S} L_{X_i}.$$

Theorem : $BNE(G) \neq \emptyset$.
Theorem : $PSNE(G) \subset BNE(G)$.
Theorem : A private strong Nash equilibrium is a private α-core strategy.
Theorem : $PSNE(G)$ may be empty.
Definition : An allocation $x^* \in L_X$ is a **private α-core allocation** for the $\mathcal{E}$ if

(1) $\sum_{i\in I} x^*_i(\omega) = \sum_{i\in I} e_i(\omega),\ \mu\text{-}a.e.$,

(2) there does not exist a coalition S and $(x_i)_{i\in S} \in \Pi_{i\in S} L_{X_i}$ such that

  (i) $\sum_{i\in S} x_i(\omega) = \sum_{i\in S} e_i(\omega),\ \mu\text{-}a.e.$,

  (ii) $\forall i \in S, v_i((x_i)_{i\in S}, (x'_i)_{i\in I\setminus S}) > v_i(x^*),\ \forall\, (x'_i)_{i\in S} \in \Pi_{i\in I\setminus S} L_{X_i}$.

Conclusion

- In an economy with differential information, it is reasonable to expect that an agent with even a zero initial endowment but better private information than all other agents that matters to the rest of the agents, should be able to exchange his superior private information for actual goods.
- In a Walrasian equilibrium with differential information where an agent has no initial endowment, he always ends up with zero consumption even if he has a superior information which is essential to the other agents.
- The private core is appropriate in a differential information economy in that it reward an agent with superior information that matters to the rest of the agents even though this agent has no endowment of physical goods. Furthermore it is incentive compatible.
- The coarse core does not have a problem of incentive compatibility but it is so big that there are some coarse core allocations which does not account the supriority of information.

3.4 Value Allocation with Differential Information

1. Private value allocation

For each economy with differential information $\mathcal{E}$ and each set of weights λ, we associate a game with side-payments (I, V_λ^p) according the rule :
For every coalition $S \subset I$,

$$V_\lambda^p(S) = \max_{x_i} \sum_{i \in S} \lambda_i \int u_i(\omega, x_i(\omega)) d\mu(\omega)$$

subject to

(1) $\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega)$, μ-*a.e.*

(2) x_i is $\mathcal{F}_i$-measurable for every $i \in S$.

Definition : An allocation $x : \Omega \mapsto \prod_{i \in I} X_i$ is a **private value allocation** of the economy with differential information $\mathcal{E}$ if

(1) x_i is $\mathcal{F}_i$-measurable for every $i \in I$,

(2) $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$, μ-*a.e.*,

(3) $\exists\lambda \in \mathbf{R}^{|I|} \setminus \{0\}$ such that $\lambda_i \int u_i(\omega, x_i(\omega))d\mu(\omega) = Sh_i(V_\lambda^p), \forall i \in I$ where $Sh_i(V_\lambda^p)$ is the Shapley value of agent i derived from the game (I, V_λ^p) and $Sh_i(V_\lambda^p) \geq \lambda_i \int u_i(\omega, e_i(\omega))d\mu, \forall i \in I$.

2. Coarse value allocation

For each economy with differential information $\mathcal{E}$ and each set of weights λ, we associate a game with side-payments (I, V_λ^c) according the rule :
For every coalition $S \subset I$,

$$V_\lambda^c(S) = \max_{x_i} \sum_{i \in S} \lambda_i \int u_i(\omega, x_i(\omega))d\mu(\omega)$$

subject to

(1) $\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega)$, μ-*a.e.*

(2) x_i is $\wedge_{i \in S}\mathcal{F}_i$-measurable for every $i \in S$.

Definition : An allocation $x : \Omega \mapsto \prod_{i \in I} X_i$ is a **coarse value allocation** of the economy with differential information $\mathcal{E}$ if

(1) x_i is $\wedge_{i \in I}\mathcal{F}_i$-measurable for every $i \in I$,

(2) $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$, μ-*a.e.*,

(3) $\exists\lambda \in \mathbf{R}^{|I|} \setminus \{0\}$ such that $\lambda_i \int u_i(\omega, x_i(\omega))d\mu(\omega) = Sh_i(V_\lambda^c), \forall i \in I$ where $Sh_i(V_\lambda^c)$ is the Shapley value of agent i derived from the game (I, V_λ^c).

3. Fine value allocation

For each economy with differential information $\mathcal{E}$ and each set of weights λ, we associate a game with side-payments (I, V_λ^f) according the rule :
For every coalition $S \subset I$,

$$V_\lambda^f(S) = \max_{x_i} \sum_{i \in S} \lambda_i \int u_i(\omega, x_i(\omega))d\mu(\omega)$$

subject to

(1) $\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega)$, μ-*a.e.*

(2) x_i is $\vee_{i \in S}\mathcal{F}_i$-measurable for every $i \in S$.

Definition : An allocation $x : \Omega \mapsto \prod_{i \in I} X_i$ is a **fine value allocation** of the economy with differential information $\mathcal{E}$ if

(1) x_i is $\vee_{i \in I} \mathcal{F}_i$-measurable for every $i \in I$,

(2) $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$, μ-*a.e.*,

(3) $\exists \lambda \in \mathbf{R}^{|I|} \setminus \{0\}$ such that $\lambda_i \int u_i(\omega, x_i(\omega)) d\mu(\omega) = Sh_i(V_\lambda^f), \forall i \in I$ where $Sh_i(V_\lambda^c)$ is the Shapley value of agent i derived from the game (I, V_λ^f) and $Sh_i(V_\lambda^f) \geq \lambda_i \int u_i(\omega, e_i(\omega)) d\mu, \forall i \in I$.

4. Strong value allocation

Definition : A feasible allocation $x : \Omega \mapsto \prod_{i \in I} X_i$ is **strongly coalitional incentive compatible** if there does not exist a coalition S and states ω, ω' with $\omega' \in E_i(\omega), \forall i \in I \backslash S$ and a net-trade vector $(z_i)_{i \in S}$ such that

(1) $\sum_{i \in S} z_i = 0$,

(2) $e_i(\omega) + (x_i(\omega') - e_i(\omega')) + z_i \in X_i, \forall i \in S$,

(3) $u_i(e_i(\omega) + (x_i(\omega') - e_i(\omega')) + z_i) > u_i(x_i(\omega)), \forall i \in S$

For each economy with differential information $\mathcal{E}$ and each set of weights λ, we associate a game with side-payments (I, V_λ^s) according to the rule :
For every coalition $S \subset I$,

$$V_\lambda^s(S) = \max_{x_i} \sum_{i \in S} \lambda_i \int u_i(\omega, x_i(\omega)) d\mu(\omega)$$

subject to

(1) $\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega)$, μ-*a.e.*

(2) x_i is strongly coalitional incentive compatible for every $i \in S$.

Definition : An allocation $x : \Omega \mapsto \prod_{i \in I} X_i$ is a **strong value allocation** of the economy with differential information $\mathcal{E}$ if

(1) x_i is strongly coalitional incentive compatible for every $i \in I$,

(2) $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega)$, μ-*a.e.*,

(3) $\exists \lambda \in \mathbf{R}^{|I|} \setminus \{0\}$ such that $\lambda_i \int u_i(\omega, x_i(\omega))d\mu(\omega) = Sh_i(V_\lambda^s), \forall i \in I$ where $Sh_i(V_\lambda^s)$ is the Shapley value of agent i derived from the game (I, V_λ^s) and $Sh_i(V_\lambda^s) \geq \lambda_i \int u_i(\omega, e_i(\omega))d\mu, \forall i \in I$.

5. Weak value allocation

Definition : A feasible allocation $x : \Omega \mapsto \prod_{i \in I} X_i$ is **weakly coalitional incentive compatible** if there does not exist a coalition S and states ω, ω' with $\omega' \in E_i(\omega), \forall i \in I \backslash S$ such that

(1) $e_i(\omega) + x_i(\omega') - e_i(\omega') \in X_i, \forall i \in S$,

(2) $u_i(e_i(\omega) + x_i(\omega') - e_i(\omega')) > u_i(x_i(\omega)), \forall i \in S$.

For each economy with differential information $\mathcal{E}$ and each set of weights λ, we associate a game with side-payments (I, V_λ^w) according to the rule :
For every coalition $S \subset I$,

$$V_\lambda^w(S) = \max_{x_i} \sum_{i \in S} \lambda_i \int u_i(\omega, x_i(\omega))d\mu(\omega)$$

subject to

(1) $\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega), \mu\text{-}a.e.$

(2) x_i is weakly coalitional incentive compatible for every $i \in S$.

Definition : An allocation $x : \Omega \mapsto \prod_{i \in I} X_i$ is a **weak value allocation** of the economy with differential information $\mathcal{E}$ if

(1) x_i is weakly coalitional incentive compatible for every $i \in I$,

(2) $\sum_{i \in I} x_i(\omega) = \sum_{i \in I} e_i(\omega), \mu\text{-}a.e.$,

(3) $\exists \lambda \in \mathbf{R}^{|I|} \setminus \{0\}$ such that $\lambda_i \int u_i(\omega, x_i(\omega)d\mu(\omega) = Sh_i(V_\lambda^w), \forall i \in I$ where $Sh_i(V_\lambda^w)$ is the Shapley value of agent i derived from the game (I, V_λ^w) and $Sh_i(V_\lambda^w) \geq \lambda_i \int u_i(\omega, e_i(\omega))d\mu, \forall i \in I$.

6. Theorems

Definition : The following are the sets of attainable utility allocations which the coalition S can attain.

(1) $U^c(S) := \{w \in \mathbf{R}^{|S|} : \exists (x_i)_{i\in S}$ such that x_i is $\wedge_{i\in S}\mathcal{F}_i$-measurable and $w_i \leq \int u_i(\omega, x_i(\omega))d\mu(\omega)$ for every $i \in S$; $\sum_{i\in S} x_i(\omega) = \sum_{i\in S} e_i(\omega), \mu\text{-}a.e.\}$.

(2) $U^f(S) := \{w \in \mathbf{R}^{|S|} : \exists (x_i)_{i\in S}$ such that x_i is $\vee_{i\in S}\mathcal{F}_i$-measurable and $w_i \leq \int u_i(\omega, x_i(\omega))d\mu(\omega)$ for every $i \in S$; $\sum_{i\in S} x_i(\omega) = \sum_{i\in S} e_i(\omega), \mu\text{-}a.e.\}$.

(3) $U(S) := \{w \in \mathbf{R}^{|S|} : \exists (x_i)_{i\in S}$ such that x_i is $\mathcal{F}_i$-measurable and

$w_i \leq \int u_i(\omega, x_i(\omega))d\mu(\omega)$ for every $i \in S$; $\sum_{i\in S} x_i(\omega) = \sum_{i\in S} e_i(\omega), \mu\text{-}a.e.\}$.

(4) $U^s(S) := \{w \in \mathbf{R}^{|S|}$: there exists strongly coalitional incentive compatible $(x_i)_{i\in S}$ such that $w_i \leq \int u_i(\omega, x_i(\omega))d\mu(\omega)\forall i \in S$ and $\sum_{i\in S} x_i(\omega) = \sum_{i\in S} e_i(\omega), \mu\text{-}a.e.\}$.

(5) $U^w(S) := \{w \in \mathbf{R}^{|S|}$: there exists weakly coalitional incentive compatible $(x_i)_{i\in S}$ such that $w_i \leq \int u_i(\omega, x_i(\omega))d\mu(\omega)\forall i \in S$ and $\sum_{i\in S} x_i(\omega) = \sum_{i\in S} e_i(\omega), \mu\text{-}a.e.\}$.

N.B. : A coarse value allocation may violate the superadditivity. Consider an economy with three agents, where the agents 1 and 2 have full information and the agent 3 has only trivial information. Then $U^c(\{3\}) \times U^c(\{1,2\}) \not\subset U^c(I)$.

Theorem : There may not exist a coarse value allocation.

Theorem : Under the assumptions, there is a fine value allocation.

Theorem : Under the assumptions, there is a private value allocation.

Lemma : If there is one commodity per state, then $U^p(S) = U^s(S) \subset U^w(S)$.

Lemma : A private value allocation is strongly coalitional incentive compatible.

7. Examples

Example 1. :

$$\begin{array}{ll} u_i(x) = \sqrt{x}, \forall i \in I, & \mu(\omega_k) = 1/4,\ \forall\, k = 1,2,3,4., \\ e_1 = (4,4,1,1), & \mathcal{F}_1 = \{\{\omega_1,\omega_2\},\{\omega_3,\omega_4\}\}, \\ e_2 = (4,1,4,1), & \mathcal{F}_2 = \{\{\omega_1,\omega_3\},\{\omega_2,\omega_4\}\}, \\ e_3 = (1,1,1,1), & \mathcal{F}_3 = \{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\}\}. \end{array}$$

1. Private value allocation

2. Core value allocation

3. Fine value allocation

4. Strong value allocation

5. Weak value allocation

Example 2. :

$$\begin{array}{ll} u_i(x_i) = \sqrt{x_i}, \forall i \in I, & \mu(\omega_k) = 1/4,\ \forall\, k = 1,2,3,4., \\ e_1 = (4,4,0,0), & \mathcal{F}_1 = \{\{\omega_1,\omega_2\},\{\omega_3,\omega_4\}\}, \\ e_2 = (4,0,4,0), & \mathcal{F}_2 = \{\{\omega_1,\omega_3\},\{\omega_2,\omega_4\}\}, \\ e_3 = (1,1,1,1), & \mathcal{F}_3 = \{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\}\}. \end{array}$$

1. **Private value allocation**
2. **Core value allocation**
3. **Fine value allocation**
4. **Strong value allocation**
5. **Weak value allocation**

Example 3. :

$$\begin{array}{ll} u_i(x) = \sqrt{x}, \forall i \in I, & \mu(\omega_k) = 1/4,\ \forall\, k = 1,2,3,4., \\ e_1 = (4,4,1,4), & \mathcal{F}_1 = \{\{\omega_1,\omega_2,\omega_4\},\{\omega_3\}\}, \\ e_2 = (4,1,4,4), & \mathcal{F}_2 = \{\{\omega_1,\omega_3,\omega_4\},\{\omega_2\}\}, \\ e_3 = (0,0,0,0), & \mathcal{F}_3 = \{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\}\}. \end{array}$$

1. **Private value allocation**
2. **Core value allocation**
3. **Fine value allocation**
4. **Strong value allocation**
5. **Weak value allocation**

Example 4. :

$$\begin{array}{ll} u_i(x) = \sqrt{x}, \forall i \in I, & \mu(\omega_k) = 1/4,\ \forall\, k = 1,2,3,4., \\ e_1 = (4,4,1,1), & \mathcal{F}_1 = \{\{\omega_1,\omega_2\},\{\omega_3,\omega_4\}\}, \\ e_2 = (4,1,4,1), & \mathcal{F}_2 = \{\{\omega_1,\omega_3\},\{\omega_2,\omega_4\}\}, \\ e_3 = (0,0,0,0), & \mathcal{F}_3 = \{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\}\}. \end{array}$$

1. **Private value allocation**
2. **Core value allocation**
3. **Fine value allocation**
4. **Strong value allocation**
5. **Weak value allocation**

Example 5. : Consider the following economy with three agents and two states that occur with equal probability.

$$
\begin{array}{lll}
v_1(x) = (\frac{1}{2}\sqrt{x(\omega_1)} + \frac{1}{2}\sqrt{x(\omega_2)}\,)^2, & e_1 = (4,0), & \mathcal{F}_1 = \{\{\omega_1\},\{\omega_2\}\} \\
v_2(x) = (\frac{1}{2}\sqrt{x(\omega_1)} + \frac{1}{2}\sqrt{x(\omega_2)}\,)^2, & e_2 = (0,4), & \mathcal{F}_2 = \{\{\omega_1\},\{\omega_2\}\} \\
v_3(x) = \frac{1}{2}x(\omega_1) + \frac{1}{2}x(\omega_2), & e_3 = (0,0), & \mathcal{F}_3 = \{\{\omega_1\},\{\omega_2\}\}
\end{array}
$$

There is a value allocation with $\lambda = (1,1,1)$ such that

$$x_1 = x_2 = (11/6, 11/6), x_3 = (2/6, 2/6)$$

Example 6. : Consider the following economy with three agents and two states that occur with equal probability.

$$
\begin{array}{lll}
v_1(x) = (\frac{1}{2}\sqrt{x(\omega_1)} + \frac{1}{2}\sqrt{x(\omega_2)}\,)^2, & e_1 = (4,0), & \mathcal{F}_1 = \{\{\omega_1\},\{\omega_2\}\} \\
v_2(x) = (\frac{1}{2}\sqrt{x(\omega_1)} + \frac{1}{2}\sqrt{x(\omega_2)}\,)^2, & e_2 = (0,4), & \mathcal{F}_2 = \{\{\omega_1\},\{\omega_2\}\} \\
v_3(x) = \frac{1}{2}x(\omega_1) + \frac{1}{2}x(\omega_2), & e_3 = (0,0), & \mathcal{F}_3 = \{\{\omega_1,\omega_2\}\}
\end{array}
$$

1. **Private value allocation**
2. **Core value allocation**
3. **Fine value allocation**
4. **Strong value allocation**
5. **Weak value allocation**

Risk aversion and information

- In examples 3 and 4, the information superiority explains that the agent 3 gets positive value allocation in spite of zero endowments and the same utility function as those of the others.

- In example 5, the risk aversion explains that the agent 3 gets positive value allocation in spite of zero endowment.

- In example 6, the agent 3 has less risk averse utility function than those of the others but he gets nothing since he has bad information.

4 On Extensive Form Implementation of Contracts in differential Information Economies

4.1 Differential information economy

We define the notion of a finite-agent economy with differential information, confining ourselves to the case where the set of states of nature, Ω, is finite and there is a finite number of goods, l, per state. $\mathcal{F}$ is a σ-algebra on Ω, I is a set of n players and R^l_+ will denote the positive orthant of R^l.

A differential information exchange economy $\mathcal{E}$ is a set $\{((\Omega,\mathcal{F}), X_i, \mathcal{F}_i, u_i, e_i, q_i) : i = 1,\ldots,n\}$ where

1. $X_i : \Omega \rightarrow 2^{R^l_+}$ is the set-valued function giving the *random consumption set* of Agent (Player) i, who is denoted also by Pi;

2. $\mathcal{F}_i$ is a partition of Ω, denoting the *private information*[9] of Pi;

3. $u_i : \Omega \times R^l_+ \rightarrow R$ is the *random utility* function of Pi;

4. $e_i : \Omega \rightarrow R^l_+$ is the *random initial endowment* of Pi, assumed to be constant on elements of $\mathcal{F}_i$, with $e_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$;

5. q_i is an $\mathcal{F}$-measurable probability function on Ω giving the *prior* of Pi. It is assumed that on all elements of $\mathcal{F}_i$ the aggregate q_i is positive. If a common prior is assumed it will be denoted by μ.

We will refer to a function with domain Ω, constant on elements of $\mathcal{F}_i$, as $\mathcal{F}_i$*-measurable*, although, strictly speaking, measurability is with respect to the σ-algebra generated by the partition. We can think of such a function as delivering information to Pi which does not permit discrimination between the states of nature belonging to any element of $\mathcal{F}_i$.

In the first period agents make contracts in the ex ante stage. In the interim stage, i.e., after they have received a signal[10] as to what is the event containing the realized state of nature, one considers the incentive compatibility of the contract.

For any $x_i : \Omega \rightarrow R^l_+$, the *ex ante expected utility* of Pi is given by

[9]Following Aumann (1987) we assume that the players' information partitions are common knowledge. Sometimes $\mathcal{F}_i$ will denote the σ-algebra generated by the partition, in which case $\mathcal{F}_i \subseteq \mathcal{F}$, as it will be clear from the context.

[10]A *signal* to Pi is an $\mathcal{F}_i$-measurable function from Ω to the set of the possible distinct observations specific to the player; that is, it induces the partition $\mathcal{F}_i$, and so gives the finest discrimination of states of nature directly available Pi.

$$v_i(x_i) = \sum_{\omega \in \Omega} u_i(\omega, x_i(\omega)) q_i(\omega). \tag{4}$$

Denote by $E_i(\omega)$ the element in the partition $\mathcal{F}_i$ which contains the realized state of nature, $\omega \in \Omega$. It is assumed that $q_i\big(E_i(\omega)\big) > 0$ for all $\omega \in \Omega$. The *interim expected utility* function of Pi is given by

$$v_i(\omega, x_i) = \sum_{\omega' \in \Omega} u_i(\omega', x_i(\omega')) q_i\big(\omega' | E_i(\omega)\big), \tag{5}$$

where

$$q_i\big(\omega' | E_i(\omega)\big) = \begin{cases} 0 & \text{for} \quad \omega' \notin E_i(\omega) \\ \frac{q_i(\omega')}{q_i\left(E_i(\omega)\right)} & \text{for} \quad \omega' \in E_i(\omega). \end{cases}$$

4.2 Private core, weak fine core, Radner equilibrium, REE and weak fine value

We define here the various equilibrium concepts in this paper, distinguishing between the free disposal and the non-free disposal case. A comparison is also made between these concepts. All definitions are in the context of the exchange economy $\mathcal{E}$ in Section 2.

We begin with some notation. Denote by $L_1(q_i, R^l)$ the space of all equivalence classes, with respect to q_i, of $\mathcal{F}$-measurable functions $f_i : \Omega \to R^l$.

L_{X_i} is the set of all $\mathcal{F}_i$-measurable selections from the random consumption set of Agent i, i.e.,

$$L_{X_i} = \{x_i \in L_1(q_i, R^l) : \ x_i : \Omega \to R^l \text{ is } \mathcal{F}_i\text{-measurable and } x_i(\omega) \in X_i(\omega) \ q_i\text{-}a.e.\}$$

and let $L_X = \prod_{i=1}^{n} L_{X_i}$.

Also let

$$\bar{L}_{X_i} = \{x_i \in L_1(q_i, R^l) : \ x_i(\omega) \in X_i(\omega) \ q_i\text{-}a.e.\}$$

and let $\bar{L}_X = \prod_{i=1}^{n} \bar{L}_{X_i}$.

An element $x = (x_1, \ldots, x_n) \in \bar{L}_X$ will be called an *allocation*. For any subset of players S, an element $(y_i)_{i \in S} \in \prod_{i \in S} \bar{L}_{X_i}$ will also be called an allocation, although strictly speaking it is an allocation to S.

We note that the above notation is employed also for purposes of comparisons with the analysis in Glycopantis - Muir - Yannelis (2001). In case there is only one good, i.e. $l = 1$, we shall use the notation $L^1_{X_i}$, $\bar{L}^1_{X_i}$ etc. When a common prior is also assumed $L_1(q_i, R^l)$ will be replaced by $L_1(\mu, R^l)$.

First we define the notion of the (ex ante) private core[11] (Yannelis (1991)).

Definition 3.1. An allocation $x \in L_X$ is said to be a *private core allocation* if

(i) $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$ and

(ii) there do not exist coalition S and allocation $(y_i)_{i\in S} \in \prod\limits_{i\in S} L_{X_i}$ such that $\sum\limits_{i\in S} y_i = \sum\limits_{i\in S} e_i$ and $v_i(y_i) > v_i(x_i)$ for all $i \in S$.

Notice that the definition above does not allow for free disposal. If the feasibility condition (i) is replaced by (i)$'$ $\sum_{i=1}^n x_i \leq \sum_{i=1}^n e_i$ then *free disposal* is allowed.

Example 3.1 Consider the following three agents economy, $I = \{1, 2, 3\}$ with one commodity, i.e. $X_i = R_+$ for each i, and three states of nature $\Omega = \{a, b, c\}$.

We assume that the initial endowments and information partitions of the agents are given by

$$e_1 = (5, 5, 0), \qquad \mathcal{F}_1 = \{\{a, b\}, \{c\}\};$$

$$e_2 = (5, 0, 5), \qquad \mathcal{F}_2 = \{\{a, c\}, \{b\}\};$$

$$e_3 = (0, 0, 0), \qquad \mathcal{F}_3 = \{\{a\}, \{b\}, \{c\}\}.$$

It is also assumed that $u_i(\omega, x_i(\omega)) = x_i^{\frac{1}{2}}$, which is a typical strictly concave and monotone function in x_i, and that every player expects that each state of nature occurs with the same probability, i.e. $\mu(\{\omega\}) = \frac{1}{3}$, for $\omega \in \Omega$. For convenience, in the discussion below expected utilities are multiplied by 3.

It was shown in Appendix II of Glycopantis - Muir - Yannelis (2001) that, without free disposal, a private core allocation of this economy is $x_1 = (4, 4, 1)$, $x_2 = (4, 1, 4)$ and $x_3 = (2, 0, 0)$. It is important to observe that in spite of the fact that Agent 3 has zero initial endowments, his superior information allows him to make a Pareto improvement for the economy as a whole and he was rewarded for doing so. In other words, Agent 3 traded his superior information for actual consumption in state a. In return Agent 3 provided insurance to Agent 1 in state c and to Agent 2 in state b. Notice that if the private information set of Agent 3 is the trivial partition, i.e., $\mathcal{F}_3^{'} = \{a, b, c\}$, then no-trade takes place and clearly in this case he gets zero utility. Thus the private core is sensitive to information asymmetries.

Next we define another core concept, the weak fine core (Yannelis (1991) and Koutsougeras - Yannelis (1993)). This is a refinement of the fine core concept of Wilson (1978). Recall

[11]The private core can also be defined as an interim concept. See Yannelis (1991) and Glycopantis - Muir - Yannelis (2001).

that the fine core notion of Wilson as well as the fine core in Koutsougeras and Yannelis may be empty in well behaved economies. It is exactly for this reason that we are working with a different concept.

Definition 3.2. An allocation $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is said to be a *weak fine core allocation* if

(i) each $x_i(\cdot)$ is $\bigvee_{i=1}^{n} \mathcal{F}_i$-measurable [12]

(ii) $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$ and

(iii) there do not exist coalition S and allocation $(y_i)_{i\in S} \in \prod_{i\in S} \bar{L}_{X_i}$ such that $y_i(\cdot) - e_i(\cdot)$ is $\bigvee_{i\in S} \mathcal{F}_i$-measurable for all $i \in S$, $\sum_{i\in S} y_i = \sum_{i\in S} e_i$ and $v_i(y_i) > v_i(x_i)$ for all $i \in S$.

Existence of private core and weak fine core allocations is discussed in Glycopantis - Muir - Yannelis (2001). The weak fine core is also an ex ante concept. As with the private core the feasibility condition can be relaxed to (ii)$'$ $\sum_{i=1}^n x_i \leq \sum_{i=1}^n e_i$. Notice however that now coalitions of agents are allowed to pool their own information and all alocations will exhaust the resource. The example below illustrates this concept.

Example 3.2 Consider the Example 3.1 without Agent 3. Then if Agents 1 and 2 pool their own information a possible allocation is $x_1 = x_2 = (5, 2.5, 2.5)$. Notice that this allocation is $\bigvee_{i=1}^{2} \mathcal{F}_i$-measurable and cannot be dominated by any coalition of agents using their pooled information. Hence it is a weak fine core allocation. [13]

Next we shall define a Walrasian equilibrium notion in the sense of Radner. In order to do so, we need the following. A *price system* is an $\mathcal{F}$-measurable, non-zero function $p : \Omega \rightarrow R^l_+$ and the *budget set* of Agent i is given by

$$B_i(p) = \{x_i : \ x_i : \Omega \rightarrow R^l \text{ is } \mathcal{F}_i\text{-measurable } \ x_i(\omega) \in X_i(\omega) \text{ and } \sum_{\omega\in\Omega} p(\omega)x_i(\omega) \leq \sum_{\omega\in\Omega} p(\omega)e_i(\omega)\}.$$

Notice that the budget constraint is across states of nature.

Definition 3.3. A pair (p, x), where p is a price system and $x = (x_1, \ldots, x_n) \in L_X$ is an allocation, is a *Radner equilibrium* if

(i) for all i the consumption function maximizes v_i on B_i

(ii) $\sum_{i=1}^n x_i \leq \sum_{i=1}^n e_i$ (free disposal), and

(iii) $\sum_{\omega\in\Omega} p(\omega) \sum_{i=1}^n x_i(\omega) = \sum_{\omega\in\Omega} p(\omega) \sum_{i=1}^n e_i(\omega)$.

Radner equilibrium is an ex ante concept. We assume free disposal, for otherwise it is well known that a Radner equilibrium with non-negative prices might not exist. This can be

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[12] $\bigvee_{i=1}^{n} \mathcal{F}_i$ denotes the smallest σ-algebra containing each $\mathcal{F}_i$.

[13] See Koutsougeras - Yannelis (1993).

seen through straightforward calculations in Example 3.1.

Next we turn our attention to the notion of REE. We shall need the following. Let $\sigma(p)$ be the smallest sub-σ-algebra of $\mathcal{F}$ for which $p : \Omega \rightarrow R^l_+$ is measurable and let $\mathcal{G}_i = \sigma(p) \vee \mathcal{F}_i$ denote the smallest σ-algebra containing both $\sigma(p)$ and $\mathcal{F}_i$. We shall also condition the expected utility of the agents on $\mathcal{G}$ which produces a random variable.

Definition 3.4. A pair (p, x), where p is a price system and $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is an allocation, is a *rational expectations equilibrium* (REE) if

(i) for all i the consumption function $x_i(\omega)$ is $\mathcal{G}_i$-measurable.

(ii) for all i and for all ω the consumption function maximizes

$$v_i(x_i|\mathcal{G}_i)(\omega) = \sum_{\omega' \in E_i^{\mathcal{G}_i}(\omega)} u_i(\omega', x_i(\omega')) \frac{q_i(\omega')}{q_i\big(E_i^{\mathcal{G}_i}(\omega)\big)}, \tag{6}$$

(where $E_i^{\mathcal{G}_i}(\omega)$ is the event in $\mathcal{G}_i$ which contains ω and $q_i(E_i^{\mathcal{G}_i}(\omega)) > 0$) subject to

$$p(\omega)x_i(\omega) \leq p(\omega)e_i(\omega)$$

i.e. the budget set at state ω, and

(iii) $\sum_{i=1}^n x_i(\omega) = \sum_{i=1}^n e_i(\omega)$ for all ω.

This is an interim concept because we condition expectations on information received from prices as well. In the definition, free disposal can easily be introduced. The idea of conditioning on the σ-algebra, $v_i(x_i|\mathcal{G}_i)(\omega)$, is rather well known.

REE can be classified as (i) *fully revealing* if the price function reveals to each agent all states of nature, (ii) *partially revealing* if the price function reveals some but not all states of nature and (iii) *non-revealing* if it does not disclose any particular state of nature.

Finally we define the concept of *weak fine value allocation* (see Krasa - Yannelis (1994)). As in the definition of the standard value allocation concept, we must first define a transferable utility (TU) game in which each agent's utility is weighted by a factor λ_i $(i = 1, ..., n)$, which allows interpersonal comparisons. In the value allocation itself no side payments are necessary.[14] A game with side payments is then defined as follows.

Definition 3.5. A game with side payments $\Gamma = (I, V)$ consist of a finite set of agents $I = \{1, ..., n\}$ and a superadditive, real valued function V defined on 2^I such that $V(\emptyset) = 0$. Each $S \subset I$ is called a coalition and $V(S)$ is the 'worth' of the coalition S.

The Shapley value of the game Γ (Shapley (1953)) is a rule that assigns to each Agent i a 'payoff, Sh_i, given by the formula[15]

[14] See Emmons - Scafuri (1985, p. 60) for further discussion.

[15] The Shapley value measure is the sum of the expected marginal contributions an agent can make to all the coalitions of which he/she is a member (see Shapley (1953)).

$$Sh_i(V) = \sum_{\substack{S \subseteq I \\ S \supseteq \{i\}}} \frac{(\mid S \mid -1)!(\mid I \mid - \mid S \mid)!}{\mid I \mid!}[V(S) - V(S \backslash \{i\})]. \tag{7}$$

The Shapley value has the property that $\sum_{i \in I} Sh_i(V) = V(I)$, i.e. it is Pareto efficient. We now define for each economy with differential information, $\mathcal{E}$, and a common prior, and for each set of weights, $\lambda_i : i = 1, \ldots, n$, the associated game with side payments (I, V_λ) (we also refer to this as a 'transferable utility' (TU) game) as follows:
For every coalition $S \subset I$ let

$$V_\lambda(S) = \max_x \sum_{i \in S} \lambda_i \sum_{\omega \in \Omega} u_i(\omega, x_i(\omega))\mu(\omega) \tag{8}$$

subject to

(i) $\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega)$, $\mu-$a.e.,

(ii) $x_i - e_i$ is $\bigvee_{i \in S} \mathcal{F}_i-$measurable.

We are now ready to define the weak fine value allocation.
Definition 3.6. An allocation $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is said to be a *weak fine value allocation* of the differential information economy, $\mathcal{E}$, if the following conditions hold
(i) Each net trade $x_i - e_i$ is $\bigvee_{i=1}^{n} \mathcal{F}_i$-measurable,
(ii) $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$ and
(iii) There exist $\lambda_i \geq 0$, for every $i = 1, ..., n$, which are not all equal to zero, with $\sum_{\omega \in \Omega} \lambda_i u_i(\omega, x_i(\omega))\mu(\omega) = Sh_i(V_\lambda)$ for all i, where $Sh_i(V_\lambda)$ is the Shapley value of Agent i derived from the game (I, V_λ), defined in (8) above.
Condition (i) requires the pooled information measurability of net trades, i. e. net trades are measurable with respect to the "join". Condition (ii) is the market clearing condition and (iii) says that the expected utility of each agent multiplied by his/her weight, λ_i must be equal to his/her Shapley value derived from the TU game (I, V_λ).
An immediate consequence of Definition 3.6 is that $Sh_i(V_\lambda) \geq \lambda_i \sum_{\omega \in \Omega} u_i(\omega, e_i(\omega))\mu(\omega)$ for every i, i.e. the value allocation is individually rational. This follows immediately from the fact that the game (V_λ, I) is superadditive for all weights λ. Similarly, efficiency of the Shapley value for games with side payments immediately implies that the value allocation is weak-fine Pareto efficient.
On the basis of the definitions and the analysis of Example 3.1 of an exhange economy with 3 agents and of Example 3.2 with 2 agents we make comparisons between the various

equilibrium notions. The calculations of all, cooperative and noncooperative, equilibrium allocations are straightforward.

Contrary to the private core any rational expectation Walrasian equilibium notion, such as Radner equilibrium or REE, will always give zero to an agent who has no initial endowments. For example, in the 3-agent economy of Example 3.1, Agent 3 receives no consumption since his budget set is zero in each state. This is so irrespective of whether his private information is the full information partition $\mathcal{F}_3 = \{\{a\}, \{b\}, \{c\}\}$ or the trivial partition $\mathcal{F}_3' = \{a, b, c\}$. Hence the Walrasian, competitive equilibrium ideas do not take into account the informational superiority of an agent.

The set of Radner equilibrium allocations, with and without free disposal, are a subset of the corresponding private core allocations. Of course it is possible that a Radner equilibrium allocation might not exist. In the two-agent economy of Example 3.2, assuming non-free disposal the unique private core is the initial endowments allocation while no Radner equilibrium exists. On the other hand, assuming free disposal, for the same example, the REE coincides with the initial endowments allocation which does not belong to the private core. It follows that *the REE allocations need not be in the private core.*

We also have that *a REE need not be a Radner equilibrium.* In Example 3.2, without free disposal no Radner equilibrium with non-negative prices exists but REE does. It is unique and it implies no-trade.

As for *the comparison between private and weak fine core allocations* the two sets could intersect but there is no definite relation. Indeed the measurability requirement of the private core allocations separates the two concepts. In Example 3.2 the allocation (5, 2.5, 2.5) to Agent 1 and (5, 2.5, 2.5) to Agent 2, as well as (6, 3, 3) and (4, 2, 2) belong to the weak fine core but not to the private core. There are many weak fine core allocations which do not satisfy the measurability condition.

For $n = 2$ one can easily verify that the weak fine value belongs to the weak fine core. However it is known (see for example Scafuri - Yannelis (1984)) that for $n \geq 3$ *a value allocation may not be a core allocation, and therefore may not be a Radner equilibrium.*

Also, in Example 3.1 *a private core allocation is not necessarily in the weak fine core.* Indeed the division (4, 4, 1), (4, 1, 4) and (2, 0, 0), to Agents 1, 2 and 3 respectively, is a private core but not a weak fine core allocation. The first two agents can get together, pool their information and do better. They can realize the weak fine core allocation, (5, 2.5, 2.5), (5, 2.5, 2.5) and (0, 0, 0) which does not belong to the private core.

Finally notice that even with free disposal no allocation which does not distribute the total resource could be in the weak fine core. The three agents can get together, distribute the surplus and increase their utility.

In the next section we shall discuss whether core and Walrasian type allocations have certain desirable properties from the point of view of incentive compatibility. Following this, we shall turn our attention in later sections to the implementation of such allocations.

4.3 Incentive compatibility

The basic idea is that an allocation is incentive compatible if no coalition can misreport the realized state of nature to the complementary set of agents and become better off.
Let us suppose we have a coalition S, with members denoted by i, and the complementary set $I \setminus S$ with members j. Let the realized state of nature be ω^*. A member $i \in S$ sees $E_i(\omega^*)$. Obviously not all $E_i(\omega^*)$ need be the same, however all Agents i know that the actual state of nature could be ω^*.
Consider now a state of nature $\omega^{'}$ with the following property. For all $j \in I \setminus S$ we have $\omega^{'} \in E_j(\omega^*)$ and for at least one $i \in S$ we have $\omega^{'} \notin E_i(\omega^*)$ (otherwise $\omega^{'}$ would be indistinguishable from ω^* for all players and, by redefining utilities appropriately, could be considered as the same element of Ω). Now the coalition S decides that each member i will announce that she has seen her own set $E_i(\omega^{'})$ which, of course, definitely contains a lie. On the other hand we have that $\omega^{'} \in \bigcap_{j \notin S} E_j(\omega^*)$ (we also denote $j \in I \setminus S$ by $j \notin S$).
Now the idea is that if all members of $I \setminus S$ believe the statements of the members of S then each $i \in S$ expects to gain. For *coalitional Bayesian incentive compatibility* (CBIC) of an allocation we require that this is not possible. This is the incentive compatibility condition used in Glycopantis - Muir - Yannelis (2001) where we gave a formal definition. We showed there that in the three-agent economy without free disposal the private core allocation $x_1 = (4, 4, 1)$, $x_2 = (4, 1, 4)$ and $x_3 = (2, 0, 0)$ is incentive compatible. This follows from the fact that Agent 3 who would potentially cheat in state a has no incentive to do so. It has been shown in Koutsougeras - Yannelis (1993) that if the utility functions are monotone and continuous then private core allocations are *always* CBIC.
On the other hand the weak fine core allocations are not always incentive compatible, as the proposed redistribution $x_1 = x_2 = (5, 2.5, 2.5)$ in the two-agent economy shows. Indeed, if Agent 1 observes $\{a, b\}$, he has an incentive to report c and Agent 2 has an incentive to report b when he observes $\{a, c\}$.
CBIC coincides in the case of a two-agent economy with *Individually Bayesian Incentive Compatibility* (IBIC) which corresponds to the case in which S is a singleton.
The concept of *Transfer Coalitionally Bayesian Incentive Compatible* (TCBIC) allocations, used in this paper[16], allows for transfers between the members of a coalition, and is

[16]see Krasa - Yannelis (1994) and Hahn - Yannelis (1997) for related concepts.

therefore a strengthening of the concept of Coalitionally Bayesian Incentive Compatibility (CBIC).

Definition 4.1. An allocation $x = (x_1, \ldots, x_n) \in \bar{L}_X$, with or without free disposal, is said to be *Transfer Coalitionally Bayesian Incentive Compatible* (TCBIC) if it is not true that there exists a coalition S, states ω^* and $\omega^{'}$, with ω^* different from $\omega^{'}$ and $\omega^{'} \in \bigcap_{i \notin S} E_i(\omega^*)$ and a random net-trade vector, z, among the members of S,

$$(z_i)_{i \in S}, \sum_S z_i = 0$$

such that for all $i \in S$ there exists $\bar{E}_i(\omega^*) \subseteq Z_i(\omega^*) = E_i(\omega^*) \cap (\bigcap_{j \notin S} E_j(\omega^*))$, for which

$$\sum_{\omega \in \bar{E}_i(\omega^*)} u_i(\omega, e_i(\omega) + x_i(\omega^{'}) - e_i(\omega^{'}) + z_i) q_i\big(\omega | \bar{E}_i(\omega^*)\big) > \sum_{\omega \in \bar{E}_i(\omega^*)} u_i(\omega, x_i(\omega)) q_i\big(\omega | \bar{E}_i(\omega^*)\big). \tag{9}$$

Notice that the z_i' s above are not necessarily measurable. The definition is cast in terms of all possible z_i' s. It follows that $e_i(\omega) + x_i(\omega^{'}) - e_i(\omega^{'}) + z_i(\omega) \in X_i(\omega)$ is not necessarily measurable. The definition means that no coalition can form with the possibility that by misreporting a state, every member will become better off if the announcement is believed by the members of the complementary set.

Returning to Definition 4.1, one then can define CBIC to correspond to $z_i = 0$ and then IBIC to the case when S is a singleton. Thus we have (not IBCI) $\Rightarrow$ (not CBIC) $\Rightarrow$ (not TCBIC). It follows that TCBIC $\Rightarrow$ CBIC $\Rightarrow$ IBIC.

We now provide a *characterization* of TCBIC:

Proposition 4.1. Let $\mathcal{E}$ be a one-good differential information economy as described above, and suppose each agent's utility function, $u_i = u_i(\omega, x_i(\omega))$ is monotone in the elements of the vector of goods x_i, that $u_i(., x_i)$ is $\mathcal{F}_i$-measurable in the first argument, and that an element $x = (x_1, \ldots, x_n) \in \bar{L}^1_X$ is a feasible allocation in the sense that $\sum_{i=1}^n x_i(\omega) = \sum_{i=1}^n e_i(\omega)\ \forall \omega$. Consider the following conditions:

(i) $x \in L^1_X = \prod_{i=1}^{n} L^1_{X_i}$ and

(ii) x is TCBIC.

Then (i) is equivalent to (ii).

Proof. First we show that (i) implies (ii) by showing that (i) and the negation of (ii) lead to a contradiction.

Let $x \in L_X$ and suppose that it is not TCBIC. Then, varying the notation for states to emphasize that Definition 4.1 does not hold, there exists a coalition S, states a and b, with

$a \neq b$ and $b \in \bigcap_{i \notin S} E_i(a)$ and a net-trade vector, z, among the members of S,

$$(z_i)_{i \in S}, \quad \sum_S z_i = 0$$

such that for all $i \in S$ there exists $\bar{E}_i(a) \subseteq Z_i(a) = E_i(a) \cap (\bigcap_{j \notin S} E_j(a))$, for which

$$\sum_{c \in \bar{E}_i(\alpha)} u_i(c, e_i(c) + x_i(b) - e_i(b) + z_i) q_i\big(c|\bar{E}_i(a)\big) > \sum_{c \in \bar{E}_i(a)} u_i(c, x_i(c)) q_i\big(c|\bar{E}_i(a)\big). \tag{10}$$

For $c \in \bar{E}_i(a), e_i(c) = e_i(a)$ since e_i is $\mathcal{F}_i$-measurable, so

$$e_i(c) + x_i(b) - e_i(b) + z_i = e_i(a) + x_i(b) - e_i(b) + z_i$$

and hence also

$$u_i\big(c, e_i(c) + x_i(b) - e_i(b) + z_i\big) = u_i\big(a, e_i(a) + x_i(b) - e_i(b) + z_i\big),$$

by the assumed $\mathcal{F}_i$-measurability of u_i.
Since, by (i), $x_i(c) = x_i(a)$ for $c \in \bar{E}_i(a)$, we similarly have $u_i\big(c, x_i(c)\big) = u_i\big(a, x_i(a)\big)$. Thus in equation (10) the common utility terms can be lifted outside the summations giving

$$u_i\big(a, e_i(a) + x_i(b) - e_i(b) + z_i\big) > u_i\big(a, x_i(a)\big)$$

and hence $e_i(a) + x_i(b) - e_i(b) + z_i > x_i(a)$, by monotonicity of u_i.
Consequently,

$$\sum_{i \in S} \big(x_i(b) - e_i(b)\big) > \sum_{i \in S} \big(x_i(a) - e_i(a)\big). \tag{11}$$

On the other hand for $i \notin S$ we have $x_i(b) - e_i(b) = x_i(a) - e_i(a)$ from which we obtain

$$\sum_{i \notin S} \big(x_i(b) - e_i(b)\big) = \sum_{i \notin S} \big(x_i(a) - e_i(a)\big). \tag{12}$$

Taking equations (11),(12) together we have

$$\sum_{i \in I} \big(x_i(b) - e_i(b)\big) > \sum_{i \in I} \big(x_i(a) - e_i(a)\big), \tag{13}$$

which is a contradiction since both sides are equal to zero, by feasibility.[17]

[17] Koutsougeras - Yannelis (1993) and Krasa - Yannelis (1994) show that (i) implies (ii) for any number of goods, but for ex post utility functions. This means that the contract is made ex ante and after the state of nature is realized we see that we have incentive compatibility. Hahn - Yannelis (1997) show that (i) implies (ii) for any number of goods and for interim utility functions. Notice that since the non-free disposal Radner equilibrium is a subset of the non-free disposal ex ante private core, it follows from Hahn - Yannelis that the non-free disposal Radner equilibrium is TCBIC.

We now show that (ii) implies (i). For suppose not. Then there exists some Agent j and states a, b with $b \in E_j(a)$ such that $x_j(a) \neq x_j(b)$. Without loss of generality, we may assume that $x_j(a) > x_j(b)$. Since $e_j(.)$ is $\mathcal{F}_j$-measurable $e_j(b) = e_j(a)$ and therefore

$$x_j(a) - e_j(a) > x_j(b) - e_j(b). \tag{14}$$

Let $S = I \backslash \{j\}$. From the feasibility of x and (14) it follows that

$$\sum_{i \in S} \big(x_i(a) - e_i(a)\big) = -\big(x_j(a) - e_j(a)\big) < -\big(x_j(b) - e_j(b)\big) = \sum_{i \in S} \big(x_i(b) - e_i(b)\big). \tag{15}$$

From (15) we have that

$$\delta = \sum_{i \in S} \big(e_i(a) + x_i(b) - e_i(b) - x_i(a)\big) > 0. \tag{16}$$

For each $i \in S$ let

$$z_i = x_i(a) - e_i(a) - x_i(b) + e_i(b) + \frac{\delta}{n-1}.$$

so that $\sum_{i \in S} z_i = 0$ and

$$e_i(a) + x_i(b) - e_i(b) + z_i > x_i(a).$$

By monotonicity of u_i, we can conclude that

$$u_i(a, e_i(a) + x_i(b) - e_i(b) + z_i) > u_i(a, x_i(a)), \tag{17}$$

for all $i \in S$, a contradiction to the fact that x is TCBIC as the role of $\bar{E}_i$ in the definition can be played by $\{a\}$.

Finally note that a particular case of $\mathcal{F}_i$-measurability of u_i is when it is independent of ω. This completes the proof of Proposition 4.1.

In the lemma that follows we refer to CBIC, as TCBIC does not make much sense since z_i is not available. CBIC is obtained when all z_i's are set equal to zero.

Lemma 4.1. Under the conditions of the Proposition, if there are only two agents then (ii) x is CBIC, which is the same as IBIC, implies (i).

Proof: For suppose not. Then lack of $\mathcal{F}_i$-measurability of the allocations implies that there exist Agent j and states a, b, where $b \in E_j(a)$, such that $x_j(b) < x_j(a)$ and therefore

$$x_j(b) - e_j(b) < x_j(a) - e_j(a). \tag{18}$$

Feasibility implies

$$x_i(b) - e_i(b) + x_j(b) - e_j(b) = x_i(a) - e_i(a) + x_j(a) - e_j(a) \tag{19}$$

from which we obtain

$$x_i(b) - e_i(b) > x_i(a) - e_i(a). \tag{20}$$

By monotonicity and the one-good per state assumption it follows that,

$$u_i(a, e_i(a) + x_i(b) - e_i(b)) > u_i(a, x_i(a)). \tag{21}$$

This implies that we have

$$u_i(a, e_i(c) + x_i(b) - e_i(b)) > u_i(a, x_i(c)) \tag{22}$$

which contradicts the assumption that x is CBIC. This completes the proof of the lemma.
The above results characterize TCBIC and CBIC in terms of private individual measurability, i.e. $\mathcal{F}_i$-measurability, of allocations. These results will enable us to conclude whether or not, in case of non-free disposal, any of the solution concepts, i.e. Radner equilibrium, REE, private core, weak fine core and weak fine value will be TCBIC whenever feasible allocations are $\mathcal{F}_i$-measurable.
It follows from the lemma that the redistribution shown in the matrix below, which is a weak fine core allocation of Example 3.2, where the ith line refers to Player i and the columns from left to right to states a, b and c,

$$\begin{pmatrix} 5 & 2.5 & 2.5 \\ 5 & 2.5 & 2.5 \end{pmatrix}$$

is not CBIC as it is not $\mathcal{F}_i$-measurable. Thus, *a weak fine core allocation may not be* CBIC.
On the other hand the proposition implies that, in Example 3.2, the no-trade allocation

$$\begin{pmatrix} 5 & 5 & 0 \\ 5 & 0 & 5 \end{pmatrix}$$

is incentive compatible. This is a non-free disposal REE, and a private core allocation.
We note that the Proposition 4.1 refers to non-free disposal. As a matter of fact Proposition 4.1 is not true if we assume free disposal. Indeed if free disposal is allowed $\mathcal{F}_i$-measurability PBE *does not* imply incentive compatibility.
In the case with *free disposal, private core and Radner equilibrium need not be incentive compatible.* In order to see this we notice that in Example 3.2 the (free disposal) Radner equilibrium is $x_1 = (4, 4, 1)$ and $x_2 = (4, 1, 4)$. The above allocation is clearly $\mathcal{F}_i$-measurable and it can easily be checked that it belongs to the (free disposal) private

core. However it is not TBIC since if state a occurs Agent 1 has an incentive to report state c and gain.

Now in employing game trees in the analysis, as it is done below, we will adopt the definition of IBIC. The equilibrium concept employed will be that of PBE. The definition of a play of the game is a directed path from the initial to a terminal node.

In terms of the game trees, a core allocation will be IBIC if there is a profile of optimal behavioral strategies and equilibrium paths along which no player misreports the state of nature he has observed. This allows for the possibility, as we shall see later, that such strategies could imply that players have an incentive to lie from information sets which are not visited by an optimal play.

In view of the analysis in terms of game trees we comment again on the general idea of CBIC. First we look at it once more, in a similar manner to the one in the beginning of Section 4.

Suppose the true state of nature is $\bar{\omega}$. Any coalition can only see that the state lies in $\bigcap\limits_{i \in S} E_i(\bar{\omega})$ when they pool their observations. If they decide to lie they must first guess at what is the true state and they will do so at some $\omega^* \in \bigcap\limits_{i \in S} E_i(\bar{\omega})$. Then of course we have $\bigcap\limits_{i \in S} E_i(\bar{\omega}) = \bigcap\limits_{i \in S} E_i(\omega^*)$. Having decided on ω^* as a possible true state, they now pick some $\omega^{'} \in \bigcap\limits_{j \notin S} E_j(\omega^*)$ and (assuming the system is not CBIC) they hope, by announcing (each of them) that they have seen $E_i(\omega^{'})$ to secure better payoffs.

This is all contingent on their being believed by $I \setminus S$. This, in turn, depends on their having been correct in their guessing that $\omega^* = \bar{\omega}$, in which case they might be believed. If $\omega^* \neq \bar{\omega}$, i.e they guess wrongly, then since $\bigcap\limits_{j \notin S} E_j(\omega^*) \neq \bigcap\limits_{j \notin S} E_j(\bar{\omega})$ they may be detected in their lie, since possibly $\omega^{'} \notin \bigcap\limits_{j \notin S} E_j(\bar{\omega})$.

This is why the definition of CBIC can only be about possible existence of situations where a lie might be beneficial. It is not concerned with what happens if the lie is detected. On the other hand the extensive form forces us to consider that alternative. It requires statements concerning earlier decisions by other players to lie or tell the truth and what payoffs will occur whenever a lie is detected, through observations or incompatibility of declarations. Only in this fuller description can players really make a decision whether to risk a lie, since only then can they balance the gains from not being caught against a definitely declared payoff if they are.

The issue is whether cooperative and noncooperative static solutions can be obtained as perfect Bayesian or sequential equilibria. That is whether such allocations can also be supported through an appropriate noncooperative solution concept. The analysis below shows that CBIC allocations can be supported by a PBE while lack of incentive compatibility implies non-support, in the sense that the two agents, left on their own, do not sign the contract. It is also shown how implementation of allocations becomes possible through the introduction in the analysis of an exogenous third party or an endogenous intermediary.

4.4 Non-implementation of free disposal private core and Radner equilibria, and of weak fine core allocations

The main point here is that lack of IBIC implies that the two agents based on their information cannot sign a proposed contract because both of them have an incentive to cheat the other one and benefit. Indeed PBE leads to no-trade. This so irrespective of whether in state a the contract specifies that they both get 5 or 4.

Note that to impose free disposal in state a causes certain problems, because the question arises as to who will check that the agents have actually thrown away 1 unit. In general, free disposal is not always a very satisfactory assumption in differential information economies with monotone preferences.

We shall investigate the possible implementation of the allocation

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \end{pmatrix}$$

in Example 3.2, contained in a contract between P1 and P2 when no third party is present. For the case with free disposal, this is both a private core and a Radner equilibrium allocation.

This allocation is not IBIC because, as we explained in the previous section, if Agent 1 observes $\{a, b\}$, he has an incentive to report c and Agent 2 has an incentive to report b when he observes $\{a, c\}$.

We construct a game tree and employ reasonable rules for describing the outcomes of combinations of states of nature and actions of the players. In fact we look at the contract

$$\begin{pmatrix} 5 & 4 & 1 \\ 5 & 1 & 4 \end{pmatrix}$$

in which the agents get as much per state as under the private core allocation above. The latter can be obtained by invoking free disposal in state a.

The investigation is through the analysis of a specific sequence of decisions and information sets shown in the game tree in Figure 1. Notice that vectors at the terminal nodes of a game tree will refer to payoffs of the players in terms of quantities. The first element will be the payoff to P1, etc.

The players are given strategies to tell the truth or to lie, i.e., we model the idea that agents truly inform each other about what states of nature they observe, or deliberately aim to mislead their opponent. The issue is what type of behavior is optimal and therefore whether a proposed contract will be signed or not. We find that the optimal strategies of the players imply no-trade.

Figures 1 and 2 show that the allocation $(5, 4, 1)$ and $(5, 1, 4)$ will be rejected by the players. They prefer to stay with their initial endowments and will not sign the proposed contract as it offers to them no advantage.

In Figure 1, nature chooses states a, b or c with equal probabilities. This choice is flashed on a screen which both players can see. P1 cannot distinguish between a and b, and P2 between a and c . This accounts for the information sets I_1, I_2 and $I_2^{'}$ which have more than one node. A player to which such an information set belongs cannot distinguish between these nodes and therefore his decisions are common to all of them. A behavioral strategy of a player is to declare which choices he would make, with what probability, from each of his information sets. Indistinguishable nodes imply the $\mathcal{F}_i$-measurability of decisions.

P1 moves first and he can either play $A_1 = \{a, b\}$ or $c_1 = \{c\}$, i.e., he can say "I have seen $\{a, b\}$ or "I have seen c". Of course only one of these declarations will be true. Then P2 is to respond saying that the signal he has seen on the screen is $A_2 = \{a, c\}$ or that it is $b_2 = \{b\}$. Obviously only one of these statements is true.

Strictly speaking the notation for choices should vary with the information set but there is no danger of confusion here. Finally notice that the structure of the game tree is such that when P2 is to act he knows exactly what P1 has chosen.

Next we specify the *rules* for calculating the payoffs, i.e. the terms of the contract:

(i) If the declarations by the two players are incompatible, that is (c_1, b_2) then no-trade takes place and the players retain their initial endowments. That is the case when either state c, or state b occurs and Agent 1 reports state c and Agent 2 state b. In state a both agents can lie and the lie cannot be detected by either of them. They are in the events $\{a, b\}$ and $\{a, c\}$ respectively, they get 5 units of the initial endowments and again they are not willing to cooperate. Therefore whenever the declarations are incompatible, no trade takes place and the players retain their initial endowments.

(ii) If the declarations are (A_1, A_2) then even if one of the players is lying, this cannot be detected by his opponent who believes that state a has occured and both players have received endowment 5. Hence no-trade takes place.

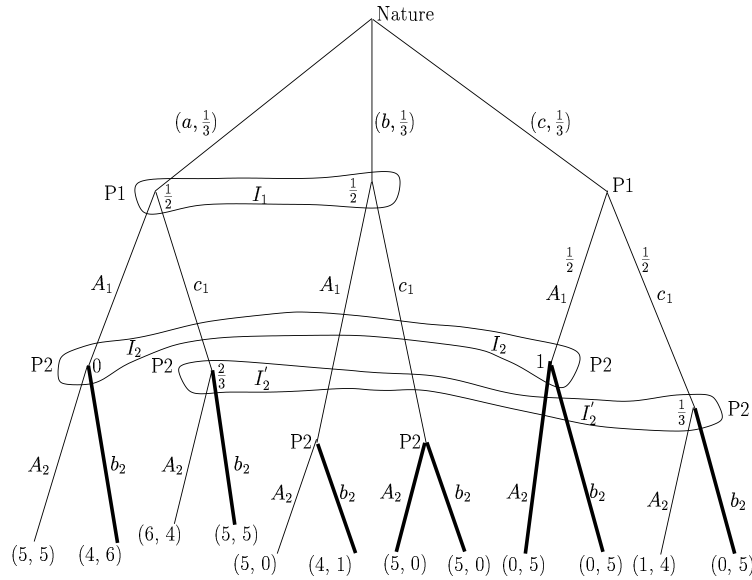
(iii) If the declarations are (A_1, b_2) then a lie can be beneficial and undetected. P1 is trapped and must hand over one unit of his endowment to P2. Obviously if his initial

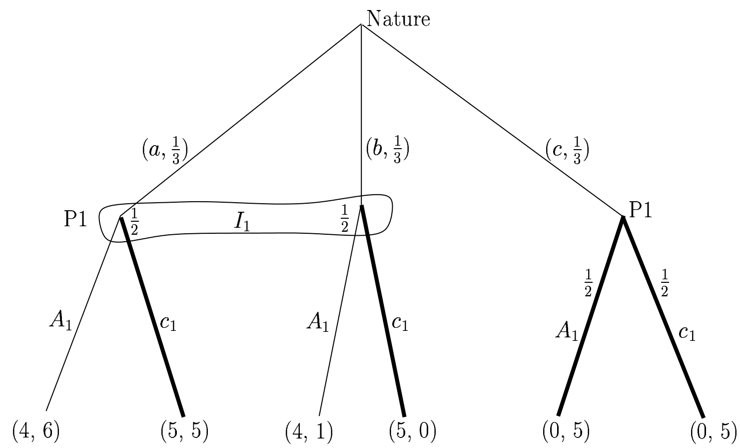
endowment is zero then he has nothing to give.
(iv) If the declarations are (c_1, A_2) then again a lie can be beneficial and undetected. P2 is now trapped and must hand over one unit of his endowment to P1. Obviously if his initial endowment is zero then he has nothing to give.
The calculations of payoffs do not require the revelation of the actual state of nature. Optimal decisions will be denoted by a heavy line. We could assume that a player does not lie if he cannot get a higher payoff by doing so.
Assuming that each player chooses optimally from his information sets, the game in Figure 1 folds back to the one in Figure 2. Inspection of Figure 1 reveals that from the information set I_2 agent P2 can play b_2 with probability 1. (A heavy line A_2 indicates that this choice also would not affect the analysis). This accounts for the payoff $(4, 6)$ and the first payoff $(0, 5)$ from left to right in Figure 2. Similarly by considering the optimal decisions from all other information sets of P2 we arrive at Figure 2. Analyzing this figure we obtain the optimal strategies of P1.
In conclusion, the optimal behavioral strategy for P1 is to play c_1 with probability 1 from I_1, i.e to lie, and from the singleton to play any probability mixture of options, and we have chosen $(A_1, \frac{1}{2}; c_1, \frac{1}{2})$. The optimal strategy of P2 is to play b_2 from both I_2 and $I_2^{'}$, i.e. to lie, and from the singletons he can either tell the truth or lie, or spin a wheel, divided in proportions corresponding to A_1 and c_1, to decide what to choose.
In Figures 1 and 2, the fractions next to the nodes in the information sets correspond to beliefs of the agents obtained, wherever possible, through Bayesian updating. I.e., they are consistent with the choice of a state by nature and the optimal behavioral strategies of the players. This means that strategies and beliefs satisfy the conditions of a PBE.
These probabilities are calculated as follows. From left to right, we denote the nodes in I_1 by j_1 and j_2, in I_2 by n_1 and n_2 and in $I_2^{'}$ by n_3 and n_4. Given the choices by nature, the strategies of the players described above and using the Bayesian formula for updating beliefs we can calculate, for example, the conditional probabilities

$$Pr(n_1/A_1) = \frac{Pr(A_1/n_1) \times Pr(n_1)}{Pr(A_1/n_1) \times Pr(n_1) + Pr(A_1/n_2) \times Pr(n_2)} = \frac{1 \times 0}{1 \times 0 + 1 \times \frac{1}{3} \times \frac{1}{2}} = 0 \tag{23}$$

and

$$Pr(n_3/c_1) = \frac{Pr(c_1/n_3) \times Pr(n_3)}{Pr(c_1/n_3) \times Pr(n_3) + Pr(c_1/n_4) \times Pr(n_4)} = \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + 1 \times \frac{1}{2} \times \frac{1}{3}} = \frac{2}{3}. \tag{24}$$

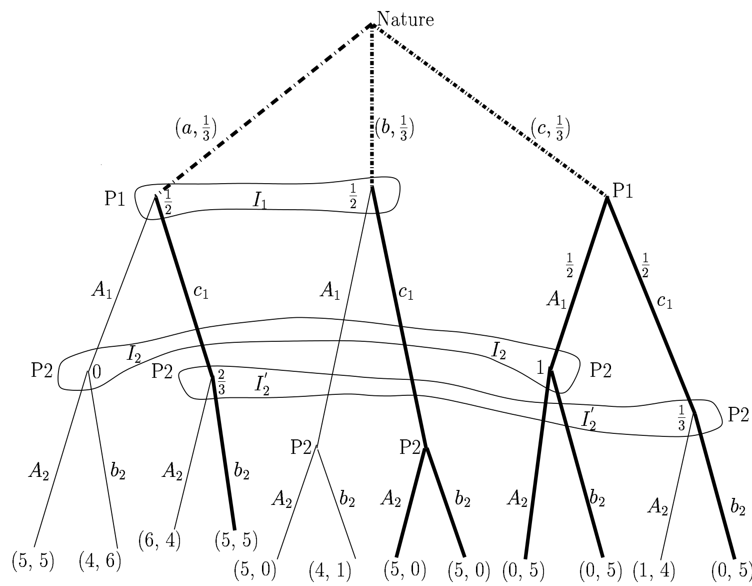




In Figure 3 we indicate, through heavy lines, plays of the game which are the outcome of the choices by nature and the optimal behavioral strategies by the players. The interrupted heavy lines signify that nature does not take an optimal decision but simply chooses among three alternatives, with equal probabilities. The directed path (a, c_1, b_2) with payoffs (5, 5) occurs with probability $\frac{1}{3}$. The paths (b, c_1, A_2) and (b, c_1, b_2) lead to payoffs (5, 0) and occur with probability $\frac{1}{3}(1-q)$ and $\frac{1}{3}q$, respectively. The values $(1-q)$ and q denote the probabilities with which P2 chooses between A_2 and b_2 from the singleton node at the end of (b, c_1). The paths (c, A_1, b_2) (c, c_1, b_2) lead to payoffs (0, 5) and occur, each, with probability $\frac{1}{3}\times\frac{1}{2}$.

For all choices by nature, at least one of the players tells a lie on the optimal play. The players by lying avoid the possibility of having to make a payment to their opponent and stay with their initial endowments. The PBE obtained above confirms the initial endowments. The decisions to lie imply that the players will not sign the contract $(5, 4, 1)$ and $(5, 1, 4)$.

We have constructed an extensive form game and employed reasonable rules for calculating payoffs and shown that the proposed allocation (5, 4, 1) and (5, 1, 4) will not be realized. A similar conclusion would have been reached if we investigated the allocation (4, 4, 1) and (4, 1, 4) which would have been brought about by considering free disposal.



Finally suppose we were to modify (iii) and (iv) of the *rules* and adopt those in Section 5 of Glycopantis - Muir - Yannelis (2001):

(iii) If the declarations are (A_1, b_2) then a lie can be beneficial and undetected, and P1 is trapped and must hand over half of his endowment to P2. Obviously if his endowment is zero then he has nothing to give.

(iv) If the declarations are (c_1, A_2) then again a lie can be beneficial and undetected. P2 is now trapped and must hand over half of his endowment to P1. Obviously if his endowment is zero then he has nothing to give.

The new rules would imply, starting from left to right, the following changes in the payoffs in Figure 1. The second vector would now be (2.5, 7.5), the third vector (7.5, 2.5), the sixth vector (2.5, 2.5) and the eleventh vector (2.5, 2.5). The analysis in Glycopantis - Muir - Yannelis (2001) shows that the weak fine core allocation in which both agents receive (5, 2.5, 2.5) cannot be implemented as a PBE. Again this allocation is not IBIC.
Since we have two agents, the weak fine value belongs to the weak fine core. We can also check through routine calculations that the non-implementable allocation $x_1 = x_2 =$

$(5, 2.5, 2.5)$ belongs to the weak fine value, with the two agents receiving equal weights. Finally we note that, in the context of Figure 1, the perfect Bayesian equilibrium implements the initial endowments allocation

$$\begin{pmatrix} 5 & 5 & 0 \\ 5 & 0 & 5 \end{pmatrix}.$$

In the case of non-free disposal, no-trade coincides with the REE and it is implementable. However as it is shown in Glycopantis - Muir - Yannelis (2002) a REE is not in general implementable.

4.5 Implementation of private core and Radner equilibria through the courts; implementation of weak fine core

We shall show here how the free disposal private core and also Radner equilibrium allocation

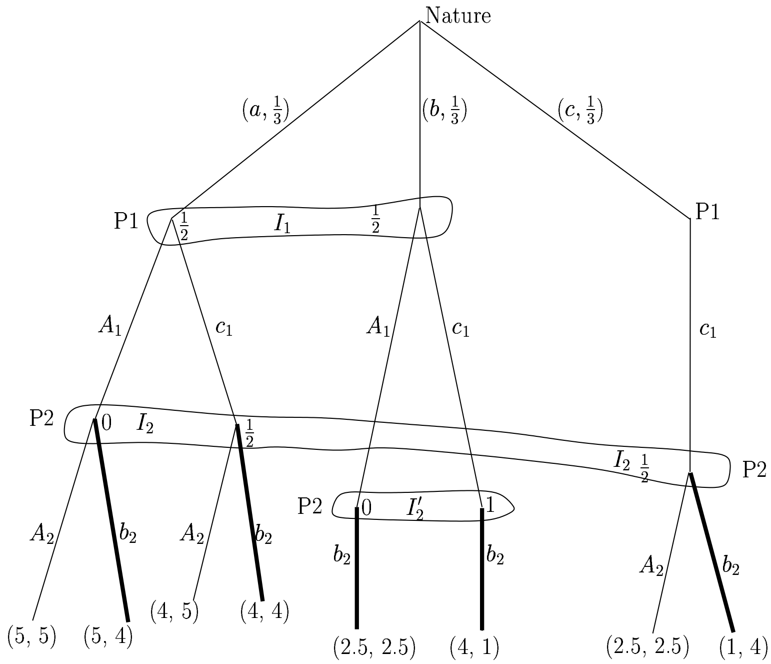
$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \end{pmatrix}$$

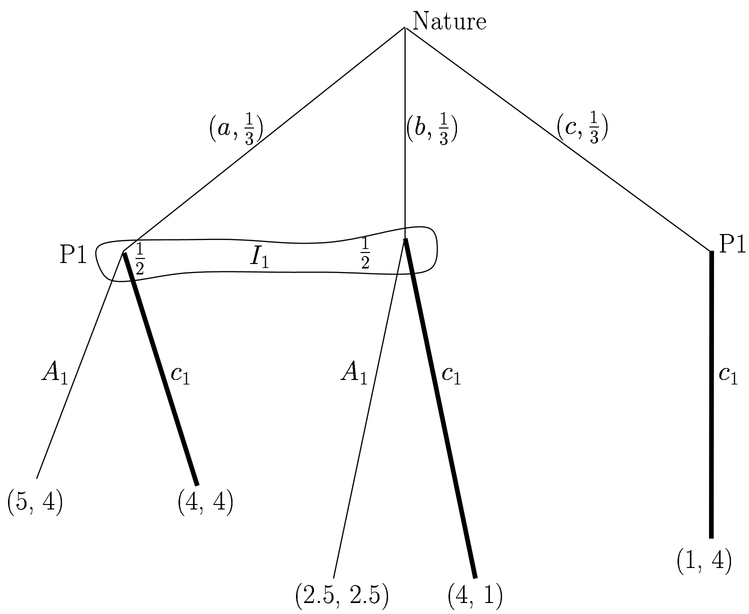
of Example 3.2 can be implemented as a PBE by invoking an exogenous third party, which can be interpreted as a court which imposes penalties when the agents lie.

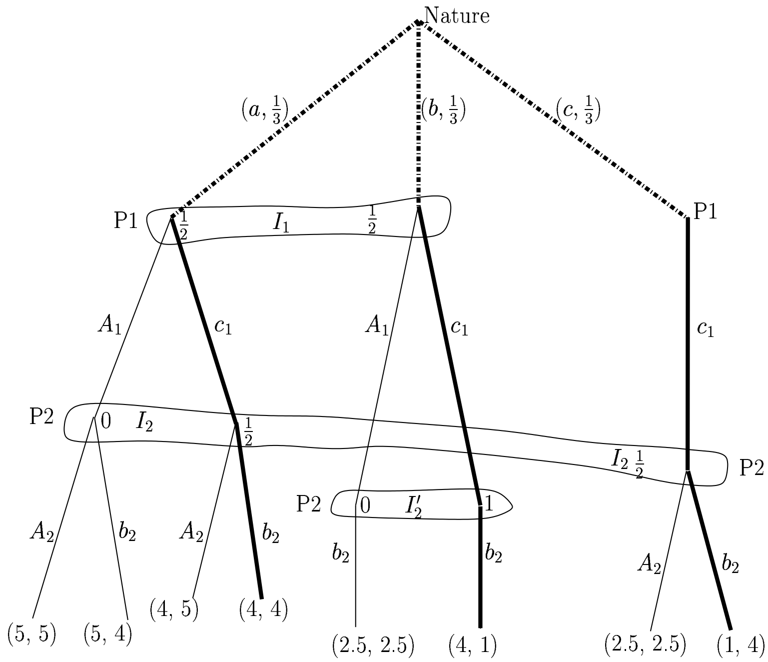
We shall assume that the agents do not hear the choice announced by the other player or that they do not pay much attention to each other because the court will verify the true state of nature.

It should be noted that now if the two players see the events (A_1, A_2) the exogenous agent will not allow them to misreport the state of nature by imposing a penalty for lying. Therefore the contract will be enforced exogenously.

The analysis is through the figures below. Figure 4 contains the information sets of the two agents, P1 and P2, their sequential decisions and the payoffs in terms of quantities. Each agent can choose either to tell the truth about the information set he is in, or to lie.


Nature
$(a, \frac{1}{3})$
$(b, \frac{1}{3})$
$(c, \frac{1}{3})$
P1
$\frac{1}{2}$
I_1
$\frac{1}{2}$
P1
A_1
c_1
A_1
c_1
c_1
P2
0
I_2
$\frac{1}{2}$
I_2 $\frac{1}{2}$
P2
P2
0
I_2'
1
A_2
b_2
A_2
b_2
b_2
b_2
A_2
b_2
(5, 5)
(5, 4)
(4, 5)
(4, 4)
(2.5, 2.5)
(4, 1)
(2.5, 2.5)
(1, 4)


Nature
$(a, \frac{1}{3})$
$(b, \frac{1}{3})$
$(c, \frac{1}{3})$
P1
$\frac{1}{2}$
I_1
$\frac{1}{2}$
P1
A_1
c_1
A_1
c_1
c_1
(5, 4)
(4, 4)
(2.5, 2.5)
(4, 1)
(1, 4)



Nature chooses states a, b and c with equal probabilities. P1 acts first and cannot distinguish between a and b. When P2 is to act he has two kinds of ignorance. Not only he cannot distinguish between a and c but also he does not know what P1 has chosen before him. This is an assumption about the relation between decisions. The one unit that the courts take from a lying agent can be considered to cover the costs of the court.

Next given the sequence of decisions of the two players, shown on the tree, we specify the rules for calculating payoffs in terms of quantities, i.e we specify the terms of the contract. They will, of course include the penalties that the court would impose to the agents for lying.

The *rules* are:

(i) If a player lies about his observation, then he is penalized by 1 unit of the good. If both players lie then they are both penalized. For example if the declarations are (c_1, b_2) and state a occurs both are penalized. If they choose (c_1, A_2) and state a occurs then the first player is penalized. If a player lies and the other agent has a positive endowment then the court keeps the quantity substracted for itself. However, if the other agent has no endowment, then the court transfers to him the one unit subtracted from the one who lied.

(ii) If the declarations of the two agents are consistent, that is (A_1, A_2) and state a occurs, (A_1, b_2) and state b occurs, (c_1, A_2) and state c occurs, then they divide equally the total endowments in the economy.

One explanation of the size of the payoffs is that if the agents decide to share, they do so voluntarily. On the other hand the court feel that they can punish them for lying but not to the extent of forcing them to share their endowments.

Assuming that each player chooses optimally, given his stated beliefs, from the information sets which belong to him, P2 chooses to play b_2 with probability 1 from both I_2 and I_2' and the game in Figure 4 folds back to the one in Figure 5. The choice of b_2 is justified as follows. We ignore for the moment the specific conditional probabilities attached to the nodes of I_2. On the other hand, starting from left to right, the sum of the probabilities of the first two nodes must be equal to $\frac{1}{2}$, and this implies that strategy b_2 overtakes, in utility terms, strategy A_2, as $\frac{1}{2}5^{\frac{1}{2}} + \frac{1}{2}2.5^{\frac{1}{2}} < 4^{\frac{1}{2}}$. It follows that P2 chooses to play the behavioral strategy b_2 with probability 1.

Now inspection of Figure 5 implies that P1 will choose c_1 from I_1. The conditional probabilities on the nodes of I_1 follow from the fact that nature chooses with equal probabilities and the optimal choice of c_1 with probability 1 follows again from the fact that $\frac{1}{2}5^{\frac{1}{2}} + \frac{1}{2}2.5^{\frac{1}{2}} < 4^{\frac{1}{2}}$.

Figure 6 indicates, through heavy lines, plays of the game which are the outcome of choices by nature and the optimal strategies of the players. The fractions next to the nodes of the information sets are obtained through Bayesian updating. I.e. they are consistent with the choice of a state by nature and the optimal behavioral strategies of the players. We have thus obtained a PBE and the above argument implies that it is unique.

The free disposal private core allocation that we are concerned with is implemented, always, by at least one of the agents lying. The reason is that they make the same move from all the nodes of an information set and the rules of the game imply that they are not eager to share their endowments. They prefer to suffer the penalty of the court.

Finally notice the following. Suppose that the penalties are changed as follows. The court is extremely severe when an agent lies while the other agent has no endowment. It takes all the endowment from the one who is lying and transfers it to the other player. Everything else stays the same. Then the game is summarized in a modified Figure 4. Numbering the end points from left to right, the 2nd vector will be replaced by (5, 0), the 3rd by (0, 5), the 4th by (0, 0), the 6th by $(0, 5)$ and the 8th one by $(5, 0)$.

The analysis of the game implies now that P2 will play A_2 from I_2 and P1 will play A_1 from I_1. Therefore invoking an exogenous agent implies that the PBE will now implement the weak fine core allocation

$$\begin{pmatrix} 5 & 2.5 & 2.5 \\ 5 & 2.5 & 2.5 \end{pmatrix}.$$

4.6 Implementation of non-free disposal private core through an endogenous intermediary

Here we draw upon the discussion in Glycopantis - Muir - Yannelis (2001) but we add the analysis that the optimal paths obtained are also part of a sequential equilibrium. Hence we obtain a stronger conclusion, in the sense that we implement the private core allocation as a sequential equilibrium, which requires more conditions than PBE.

In the case we consider now there is no court and the agents in order to decide must listen to the choices of the other players before them. The third agent, P3, is endogenous and we investigate his role in the implementation, or realization, of private core allocations.

Private core without free disposal seems to be the most satisfactory concept. The third agent who plays the role of the intermediary implements the contract and gets rewarded in state a. We shall consider the private core allocation, of Example 3.1,

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \\ 2 & 0 & 0 \end{pmatrix}.$$

We know that such core allocations are CBIC and we shall show now how they can be supported as perfect Bayesian equilibrium of a noncooperative game.

P1 cannot distinguish between states a and b and P2 between a and c. P3 sees on the screen the correct state and moves first. He can either announce exactly what he saw or he can lie. Obviously he can lie in two ways. When P1 comes to decide he has his information from the screen and also he knows what P3 has played. When P2 comes to decide he has his information from the screen and he also knows what P3 and P1 played before him. Both P1 and P2 can either tell the truth about the information they received from the screen or they can lie.

We must distinguish between the announcements of the players and the true state of nature. The former, with the players' temptations to lie, cannot be used to determine the true state which is needed for the purpose of making payoffs. P3 has a special status but he must also take into account that eventually the lie will be detected and this can affect his payoff.

The *rules* of calculating payoffs, i.e. the terms of the contract, are as follows:

If P3 tells the truth we implement the redistribution in the matrix above which is proposed for this particular choice of nature.

If P3 lies then we look into the strategies of P1 and P2 and decide as follows:

(i) If the declaration of P1 and P2 are incompatible we go to the initial endowments and each player keeps his.

(ii) If the declarations are compatible we expect the players to honour their commitments for the state in the overlap, using the endowments of the true state, provided these are positive. If a player's endowment is zero then no transfer from that agent takes place as he has nothing to give.

The extensive form game is shown in Figure 7, in which the heavy lines can be ignored in the first instance. We are looking for a PBE, i.e. a set of optimal behavioral strategies consistent with a set of beliefs. The beliefs are indicated by the probabilities attached to the nodes of the information sets, with arbitrary r, s, q, p and t between 0 and 1. The folding up of the game tree through optimal decisions by P2, then by P1 and subsequently by P3 is explained in Glycopantis - Muir - Yannelis (2001).

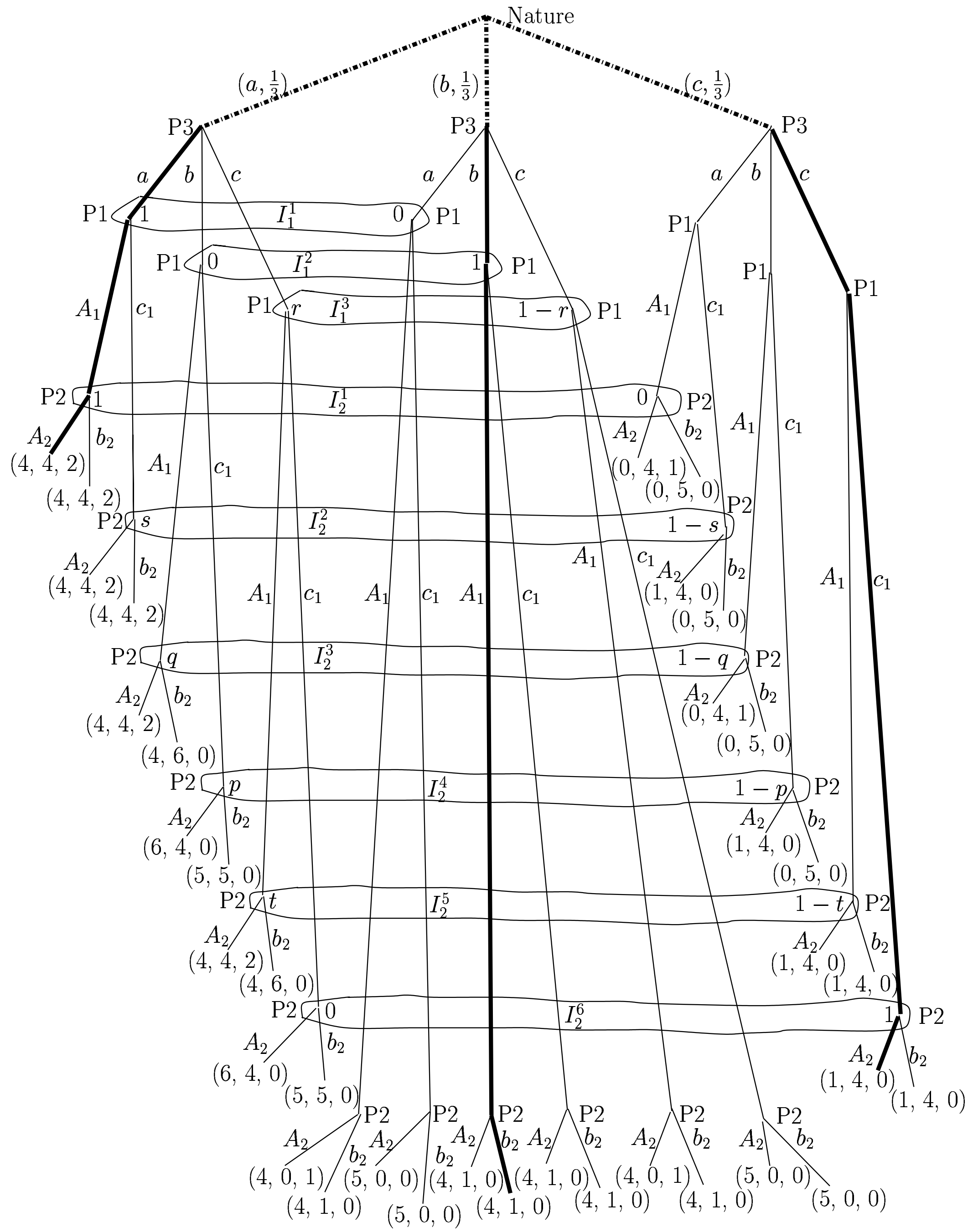
In Figure 7 we indicate through heavy lines the equilibrium paths. The interrupted heavy lines at the beginning of the tree signify that nature does not take an optimal decision but simply chooses among three alternatives, with equal probabilities. The directed paths (a, a, A_1, A_2) with payoffs (4, 4, 2), (b, b, A_1, b_2) with payoffs (4, 1, 0) and (c, c, c_1, A_2) with payoffs (1, 4, 0) occur, each, with probability $\frac{1}{3}$. It is clear that nobody lies on the optimal paths and that the proposed reallocation is incentive compatible and hence it will be realized.

Along the optimal paths nobody has an incentive to misrepresent the realized state of nature and hence the private core allocation is incentive compatible. However even optimal strategies can imply that players might have an incentive to lie from information sets which are not visited by the optimal play of the game. For example, P1, although he knows that nature has chosen a or b, has an incentive to declare c_1 from I_1^3, trying to take advantage of a possible lie by P3. Similarly P2, although he knows that nature has chosen a or c, has an incentive to declare b_2 from I_2^2, I_2^3, I_2^4 and I_2^5, trying to take advantage of possible lies by the other players. Incentive compatibility has now been defined to allow that the optimal strategies can contain lies, while there must be an optimal play which does not.

We also note that the same payoffs, i.e. (4, 4, 2), (4, 1, 0) and (1, 4, 0), can be confirmed as a PBE for all possible orders of the players.

Next we turn our attention to obtaining a sequential equilibrium. This adds further conditions to those of a PBE. Now, it is also required that the optimal behavioral strategies and the beliefs consistent with these are the limit of a sequence consisting of completely mixed behavioral strategies, that is all choices are played with positive probability, and the implied beliefs. Throughout the sequence it is only required that beliefs are consistent with the strategies. The latter are not expected to be optimal.

We discuss how the PBE shown in Figure 7 can also be obtained as a sequential equilibrium in the sense of Kreps - Wilson (1982). Therefore, we are looking for a sequence of positive probabilities attached to all the choices from each information set and beliefs consistent with these such that their limits are the results given in Figure 7.



First we specify the positive probabilities, i.e. the completely mixed strategies, with which

the players choose the available actions. The sequence is obtained through $\{n = 1, 2, ...\}$.
In the first instance we consider the singletons from left to right belonging to P3. At the first one the positive probabilities attached to the various actions are given by $(a, 1 - \frac{2}{n};\ b, \frac{1}{n};\ c, \frac{1}{n})$, at the second one by $(a, \frac{1}{n};\ b, 1 - \frac{2}{n};\ c, \frac{1}{n})$ and at the third one by $(a, \frac{1}{n};\ b, \frac{1}{n};\ c, 1 - \frac{2}{n})$.
Then we come to the probabilities with which P1 chooses his actions from the various information sets belonging to him. From I_1^1 and I_1^2 the choices and the probabilities attached to these are $(A_1, 1 - \frac{1}{n};\ c_1, \frac{1}{n})$, and from I_1^3, as well as from all the singletons, they are $(A_1, \frac{1}{n};\ c_1, 1 - \frac{1}{n})$.
With respect to P2 choices and probabilities are given as follows. From I_2^1 and I_2^6 they are $(A_2, 1 - \frac{1}{n};\ b_2, \frac{1}{n})$ and from I_2^2, I_2^3, I_2^4 and I_2^5 they are $(A_2, \frac{1}{n};\ b_2, 1 - \frac{1}{n})$. With respect to the singletons belonging to P2 we have for all of them $(A_2, \frac{1}{n};\ b_2, 1 - \frac{1}{n})$.
Beliefs are indicated by the probabilities attached to the nodes of the information sets. Below by the left (right) probability we mean the consistent with the above behavioral strategies belief that the player attaches to being at the left (right) corner node of an information set. We also give the limit of these beliefs as n tends to ∞.
In I_1^1 the left probability is $\dfrac{1 - \frac{2}{n}}{1 - \frac{1}{n}}$ and the right probability is $\dfrac{\frac{1}{n}}{1 - \frac{1}{n}}$. The limit is $(1, 0)$.
In I_1^2 the left probability is $\dfrac{\frac{1}{n}}{1 - \frac{1}{n}}$ and the right probability is $\dfrac{1 - \frac{2}{n}}{1 - \frac{1}{n}}$. The limit is $(0, 1)$.
In I_1^3 the left probability is $\dfrac{1}{2}$ and the right probability is $\dfrac{1}{2}$. The limit is $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.
In I_2^1 the left probability is $\dfrac{(1 - \frac{1}{n})(1 - \frac{2}{n})}{(1 - \frac{2}{n})(1 - \frac{1}{n}) + (\frac{1}{n})^2}$ and the right probability is $\dfrac{(\frac{1}{n})^2}{(1 - \frac{1}{n})(1 - \frac{2}{n}) + (\frac{1}{n})^2}$.
The limit is $(1, 0)$.
In I_2^2 the left probability is $\dfrac{(1 - \frac{2}{n})\frac{1}{n}}{(1 - \frac{2}{n})\frac{1}{n} + (1 - \frac{1}{n})\frac{1}{n}}$ and the right probability is $\dfrac{(1 - \frac{1}{n})\frac{1}{n}}{(1 - \frac{2}{n})\frac{1}{n} + (1 - \frac{1}{n})(\frac{1}{n})}$.
The limit is $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.
In I_2^3 the left probability is $\dfrac{(1 - \frac{1}{n})\frac{1}{n}}{(1 - \frac{1}{n})\frac{1}{n} + (\frac{1}{n})^2}$ and the right probability is $\dfrac{(\frac{1}{n})^2}{(1 - \frac{1}{n})\frac{1}{n} + (\frac{1}{n})^2}$
The limit is $(1, 0)$.
In I_2^4 the left probability is $\dfrac{(\frac{1}{n})^2}{(1 - \frac{1}{n})\frac{1}{n} + (\frac{1}{n})^2}$ and the right probability is $\dfrac{(1 - \frac{1}{n})\frac{1}{n}}{(1 - \frac{1}{n})\frac{1}{n} + (\frac{1}{n})^2}$
The limit is $(0, 1)$.
In I_2^5 he left probability is $\dfrac{(\frac{1}{n})^2}{(1 - \frac{2}{n})\frac{1}{n} + (\frac{1}{n})^2}$ and the right probability is $\dfrac{(1 - \frac{2}{n})\frac{1}{n}}{(1 - \frac{2}{n})\frac{1}{n} + (\frac{1}{n})^2}$.
The limit is $(0, 1)$.
In I_2^6 the left probability is $\dfrac{(1 - \frac{1}{n})\frac{1}{n}}{(1 - \frac{1}{n})\frac{1}{n} + (1 - \frac{1}{n})(1 - \frac{2}{n})}$ and the right probability is

$\frac{(1-\frac{1}{n})(1-\frac{2}{n})}{(1-\frac{1}{n})\frac{1}{n}+(1-\frac{1}{n})(1-\frac{2}{n})}$. The limit is $(0,1)$.
The belief attached to each singleton is that it has been reached with probability 1.
The limits of the sequence of strategies and beliefs confirm a particular Bayesian equilibrium as a sequential one. In an analogous manner, sequential equilibria can also be obtained for the models analyzed in the previous sections.

4.7 Concluding remarks

As we have already emphasized in Glycopantis - Muir - Yannelis (2001), we consider the area of incomplete and differential information and its modelling important for the development of economic theory. We believe that the introduction of game trees, which gives a dynamic dimension to the analysis, helps in the development of ideas.
The discussion in that paper is in the context of one-good examples without free disposal. The conclusion was that core notions which may not be CBIC, such as the weak fine core, cannot easily be supported as a PBE. On the other hand, in the presence of an agent with superior information, the private core which is CBIC can be supported as a PBE. The discussion provided a noncooperative interpretation or foundation of the private core while making, through the game tree, the individual decisions transparent. In this way a better understanding of how incentive compatible contracts are formed is obtained.
In the present paper we investigate, in a one-good, two-agent economy, with and without free disposal, the implementation of private core, of Radner equilibrium, of weak fine core and weak fine values allocations. We obtain, through the construction of a tree with reasonable rules, that free disposal private core allocations, to which also the Radner equilibrium belongs, are not implementable. A brief comparison of the idea of CBIC in the static presentation with the case when the analysis is in terms of game trees is made. It is surprising that free disposal destroys incentive compatibility and creates problems for implementation. Implementation in this case can be achieved by invoking an exogenous third party which can be thought of as a court that penalizes lying agents. It is of course possible that rational agents, once they realize that they can be cheated, might decide not to trade rather than rely on a third party which has to prove that he has perfect knowledge and can execute the correct trades. Notice that the third, exogenous party, in this case the court, plays the role of the mechanism designer in the relevant implementation literature (see Hahn - Yannelis (2001) and the references there).
Similarly, implementation of a private core allocation becomes possible through the introduction of an endogenous third party with zero endowments but with superior information. In this case the third party is part of the model, i.e. an agent whose superior information

allows him to play the role of an intermediary. The analysis overlaps with the one in Glycopantis - Muir - Yannelis (2001). On the other hand we show here that implementation can also be achieved through a sequential equilibrium. It should be noted that the endogenous third agent is rewarded for his superior information by receiving consumption in a particular state, in spite of the fact that he has zero initial endowments in each state. However, both Radner equilibria and REE would not recognize a special role to such an agent. These Walrasian type notions would award to him zero consumption in all states of nature.

In summary, the analysis here considers the relation between, cooperative and noncooperative, static equilibrium concepts and noncooperative, game theoretic dynamic processes in the form of game trees. We have examined the possible support and implementation as perfect Bayesian equilibria of the cooperative concepts of the private core and the weak fine core, and the noncooperative generalized, Walrasian type equilibrium notions of Radner equilibrium and REE. In effect what we are doing is to look directly into the Bayesian incentive compatibility of the corresponding allocations, as if they were contracts, and then consider their implementability.

4.8 Appendix: A note on PBE.

In this note we look briefly at equilibrium notions when sequential decisions are taken by the players, i.e. in the context of game trees. For strategies we shall employ the following idea. A *behavioral strategy* for a player being an assignment to each of his information sets of a probability distribution over the options available from that set. For a game of perfect recall, Kuhn (1953) shows that analysis of the game in terms of behavioral strategies is equivalent to that in terms of, the more familiar, mixed strategies. In any case, behavioral strategies are more natural to employ with an extensive form game. Sometimes we shall refer to them simply as *strategies*.

Consider an extensive form game and a given profile of behavioral strategies

$$s = \{s_i : i \in I\}$$

where I is the set of players.

When s is used each node of the tree is reached with probability obtained by producting the option probabilities given by s along the path leading to that node. In particular, there is a probability distribution over the set of terminal nodes so the expected payoff E_i to each player Pi may be expressed in terms of option probabilities from each information set.

Consider any single information set J owned by Pi, with corresponding option probabilities

$(1-\pi_J, \pi_J)$, where for simplicity of notation we assume binary choice. The dependence of E_i on π_J is determined only by the paths which pass through J. Taking any one of these paths, on the assumption that the game is of perfect recall, the term it contributes to E_i will only involve π_J once in the corresponding product of probabilities. Thus, on summing over all such paths, the dependence of E_i on π_J is seen to be linear, with coefficients depending on the remaining components of s.

This allows the formation of a reaction function expressing π_J in terms of the remaining option probabilities, by optimizing π_J while holding the other probabilities constant; hence the Nash equilibria are obtained, as usual, as simultaneous solutions of all these functional relations. We are here adopting an *agent* form for a player, where optimization with respect to each of his decisions is done independently from all the others. A solution is guaranteed by the usual proof of existence for Nash equilibria.

For example, consider the tree in Figure 4, denoting the option probabilities from I_1, I_2 by $(1-\alpha, \alpha), (1-\beta, \beta)$ respectively. The payoff functions are then (apart from the factor $\frac{1}{3}$ expressing the probability of Nature's choice, and leaving out terms not involving α which come from paths not passing through I_1, I_2

$$E_1 = 5(1-\alpha)(1-\beta)+5(1-\alpha)\beta+4\alpha(1-\beta)+4\alpha\beta+2.5(1-\alpha)+4\alpha+\ldots = 7.5+0.5\alpha+\ldots;$$

$$E_2 = 5(1-\alpha)(1-\beta)+4(1-\alpha)\beta+5\alpha(1-\beta)+4\alpha\beta+2.5(1-\beta)+4\beta+\ldots = 7.5+0.5\beta+\ldots.$$

Since the coefficient of α in E_1 is positive, the optimal choice of α, i.e. the reaction function of Agent 1 is 1. Similarly for β in E_2 we obtain the value 1, and this is the reaction function of Agent 2.

Note that in any such calculation, only the coefficient of each π_J is important for the optimization — the rest of E_i is irrelevant. We may similarly treat the 21 option probabilities in Figure 7, obtaining 21 conditions which they must satisfy. These are quite complex and there are, probably, many solutions but it may be checked that the one given satisfies all conditions.

When an equilibrium profile is used, it is possible that some nodes are visited with zero probability. This means that the restriction of the strategy profile to subsequent nodes has no effect on the expected payoffs, so may be chosen arbitrarily. To eliminate this redundancy in the set of Nash equilibria, a refinement of the equilibrium concept to that of perfect equilibrium, was introduced for games of perfect information — that is, games in which each information set is a singleton. This requires an equilibrium strategy also to be a Nash equilibrium for *all* sub-games of the given game. In other words, the strategy profile should be a Nash equilibrium for the game which might be started from any node of the given tree, not just the nodes actually visited in the full game.

Any attempt to extend this notion to general games encounters the problem that sub-trees might start from nodes which are not in singleton information sets. In such a case, the player who must move first cannot know for certain at which node he is located within that set. He can only proceed if he adopts *beliefs* about where he might be, in the form of a probability distribution over the nodes of the information set. Moreover, these beliefs must be common knowledge, for the other players to be able to respond appropriately, so the desired extension of the equilibrium concept must take into account both strategies and beliefs of the players. The game will be played from any information set as if the belief probabilities had been realised by an act of nature.

We need, therefore, to consider pairs (s, μ), consisting of a behavioral strategy profile s and a belief profile

$$\mu = \{\mu_J : J \in \mathcal{J}\}.$$

Here, $\mathcal{J}$ denotes the set of information sets and μ_J is a probability distribution over the nodes of information set J, expressing the beliefs of the player who might be required to play from that set. Given the belief profile, we then require that the strategy profile give a perfect equilibrium, in the sense of being optimal for each player starting from every information set. But we need also to consider the source of the beliefs.

Given any behavioral strategy profile s denote the probability of reaching any node a, using s, by $\nu(a)$. Consider first an information set, J, not all of whose nodes are visited with zero probability when using s. We may calculate the conditional probability of being at a node $a \in J$ given that it is in J by

$$\nu(a|J) = \frac{\nu(\{a\} \cap J)}{\nu(J)} = \frac{\nu(a)}{\nu(J)}$$

since $a \in J \Rightarrow \{a\} \cap J = \{a\}$. Thus the belief probabilities $\mu_J(a) = \nu(a|J)$ for J are just the relative probabilities of reaching the nodes of J.

For example, returning to Figure 4 and employing the only Nash solution $\alpha = \beta = 1$ noted above, the probabilities of reaching the nodes of I_2 are $0, \frac{1}{3}, \frac{1}{3}$ which relativises, given the condition that we reach I, to $0, \frac{1}{2}, \frac{1}{2}$ as stated.

Thus for a *PBE*, the behavioral strategy-belief profile pair (s, μ) should satisfy two conditions:

(i) For the given belief profile μ, the strategy profile s should be a perfect equilibrium, as defined above;

(ii) For the given strategy profile s, the belief profile μ should be calculated at each information set for which $\nu(I) \neq 0$ by the formula above.

Justifications of the concept of perfect equilibrium in games of perfect information will argue that the players need to have good strategies to employ, even were something to go wrong with the intended play so that the game accidentally enters sub-trees which ought not to be accessed. One way to argue this is through the notion of a trembling hand which makes errors, so possibly choosing the wrong move. Employing this same idea in the context of perfect Bayesian equilibria, we can allow small perturbations in the strategies, such that all information sets are visited with non-zero probability. Then the relation determining beliefs from strategies is well posed and we may consider only beliefs which arise as limiting cases of such perturbations. This more restrictive definition of equilibrium is called a *sequential equilibrium*.

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6 HOMEWORKS AND SOLUTIONS

HOMEWORK 1

1. Let $X_i = R_+^\ell$ be the consumption set. Let $\mathcal{B}_i : R_+^\ell \to 2^{X_i}$ be defined by

$$\mathcal{B}_i(p) = \{x_i \in X_i : p \cdot x_i \leq p \cdot e_i\},$$

where the endowment $e_i \in X_i$ and $p \in R_+^\ell$. Show the following.

(1) $\mathcal{B}_i$ is homogeneous of degree zero in $p \in R_+^\ell$.

By (1), we restrict the domain of $\mathcal{B}_i$ to the simplex $\Delta := \{p \in R_+^\ell : \sum_{k=1}^\ell p_k = 1\}$ instead of R_+^ℓ.

(2) $\mathcal{B}_i$ is nonempty-valued.

(3) $\mathcal{B}_i$ is convex-valued.

(4) $\mathcal{B}_i(p)$ is compact if $p \gg 0$.

(5) $\mathcal{B}_i$ is upper hemicontinuous.

(6) If there is some $\hat{x} \in X_i$ such that $p \cdot \hat{x} < p \cdot e_i$ for every $p \in \Delta$, then $\mathcal{B}_i$ is lower hemi-continuous.

2. Let X_i be a nonempty compact convex subset of R^ℓ for every $i \in I$ (I is a finite set). Consider the utility maximization problem as follows: For every $i \in I$,

$$\max\{u_i(x) : x \in \mathcal{B}_i(p)\}.$$

where $\mathcal{B}_i : \Delta \to 2^{X_i}$ is the correspondence defined above. Assume that u_i is continuous and quasi-concave, and $e_i \in X_i \cap R_{++}^\ell$. The solution of the program is represented by the correspondence $\varphi_i : \Delta \to 2^{X_i}$.

(1) Let $C = \sum_{i \in I} X_i$. Is C is nonempty, compact, and convex?

(2) What are the properties of the correspondence φ_i?

(3) Let $K = C - \{\sum_{i \in I} e_i\}$. Is K nonempty, compact, and convex? Define the correspondence $\Psi : \Delta \to 2^K$ by

$$\Psi(p) = \sum_{i \in I} [\varphi_i(p) - \{e_i\}].$$

What are the properties of the correspondence Ψ?

(4) Show that for each $p \in \Delta$, $p \cdot z \leq 0$ for every $z \in \Psi(p)$.

(5) Define the correspondence $\mu : K \to 2^{\Delta}$ by

$$\mu(z) = \{p \in \Delta : p \cdot z = \max_{p' \in \Delta} p' \cdot z\}.$$

Show that μ is nonempty-valued, compact-valued, convex-valued, and upper hemi-continuous.

(6) Define the correspondence $\mu \times \Psi : \Delta \times K \to 2^{\Delta \times K}$ by $(\mu \times \Psi)(p, z) = \mu(z) \times \Psi(p)$. Applying the Kakutani fixed point theorem, show that there exists $p^* \in \Delta$ such that $\Psi(p^*) \cap R^{\ell}_{-} \neq \emptyset$.

3. Let $X = R^{\ell}_{+}$. Assume that $u : X \to R$ is a continuous function. Define the correspondence $P : X \to 2^X$ by $P(x) = \{x' \in X : u(x') > u(x)\}$.

(1) Show that $P^{-1}(y) = \{x \in X : y \in P(x)\}$ is open.

(2) Using the result of (1), prove that P is lower hemi-continuous.

4. Let X be a nonempty compact convex subset of R^{ℓ}, Y be a nonempty compact convex subset of R^m, and V be an open subset of X. Let $\varphi : X \to 2^Y$ be an upper hemi-correspondence and $f : V \to Y$ be a continuous selection from $\varphi|_V$. Then the correspondence $\psi : X \to 2^Y$ defined by

$$\psi(x) = \begin{cases} \{f(x)\} & \text{if } x \in V \\ \varphi(x) & \text{if } x \notin V \end{cases}$$

is upper hemi-continuous.

5. Let $X \subset R^{\ell}$ and $Y \subset R^m$.

(1) Let a correspondence $P : X \to 2^Y$ have open graph and Y be convex. Prove that $coP : X \to 2^Y$ has open graph.

(2) Let $coP : X \to 2^Y$ have open graph and $A : X \to 2^Y$ be lower hemi-continuous. Define $\mu : X \to 2^Y$ by $\mu(x) := coP(x) \cap A(x)$. Prove that μ is lower hemi-continuous.

SOLUTION 1

1.

(1) Since $p \cdot x \le p \cdot e_i$ iff $\lambda p \cdot x \le \lambda p \cdot e_i$ for every $\lambda > 0$, $\mathcal{B}_i(p) = \mathcal{B}_i(\lambda p)$ for every $\lambda > 0$, *i.e.*, $\mathcal{B}_i$ is homogeneous of degree zero.

(2) Since $e_i \in \mathcal{B}_i(p)$ for every $p \in \Delta$, $\mathcal{B}_i$ is nonempty-valued. □

(3) Let $x \in \mathcal{B}_i(p)$ and $x' \in \mathcal{B}_i(p)$. Since R^ℓ_+ is convex, $x^\alpha = \alpha x + (1-\alpha)x' \in X_i$. And $p \cdot x^\alpha \le \alpha p \cdot x + (1-\alpha) p \cdot x' \le p \cdot e_i$. That is, $x^\alpha \in \mathcal{B}_i(p)$ so that $\mathcal{B}_i$ is convex-valued. □

(4) Since $f(x) = p \cdot x$ is a continuous function from R^ℓ_+ to R_+, $\mathcal{B}_i(p) = f^{-1}([0, p \cdot e_i])$ is closed in R^ℓ_+. Let $p_k > 0$ be the minimum of $p_1, p_2, \ldots, p_\ell$. Then, for every $x \in \mathcal{B}_i(p)$, we get

$$x_j \le \frac{p_j}{p_k} x_j \le \frac{p \cdot x}{p_k} \le \frac{p \cdot e_i}{p_k}, \ \forall j = 1, 2, \ldots, \ell.$$

Thus $0 \le x \le (p \cdot e_i / p_k, \ldots p \cdot e_i / p_k)$, *i.e.*, $\mathcal{B}_i(p)$ is bounded. Hence, $\mathcal{B}_i(p)$ is compact in R^ℓ_+. □

(5) Fix p^*. Choose any open neighborhood W in R^ℓ_+ such that contains $\mathcal{B}_i(p^*)$. Then we can find an open neighborhood $V = \{x \in R^\ell_+ : p^* \cdot x < p^* \cdot e_i + \varepsilon\}$ with $\varepsilon > 0$, which is contained in W. Now define $A = \{k \in I : p^*_k > 0\}$. Then for every $k \in A$, we have

$$\frac{p^* \cdot e_i}{p^*_k} < \frac{p^* \cdot e_i + \varepsilon}{p^*_k},$$

so that there is a $\delta > 0$ such that for every $p \in B_{\delta(p^*)} \cap \Delta$,

$$\frac{p \cdot e_i}{p_k} < \frac{p^* \cdot e_i + \varepsilon}{p^*_k}.$$

Let

$$\epsilon^k = (0, \ldots, 0, \overbrace{1}^{k\text{-th}}, 0, \ldots, 0),$$

for each $k = 1, \ldots, \ell$ and $\epsilon^0 = 0$, and for each $k \in A' = A \cup \{0\}$, let

$$\hat{x}^k = \frac{p \cdot e_i}{p_k} \epsilon^k.$$

Then we can easily verify that if $x \in \mathcal{B}_i(p)$, x can be expressed as follows:

$$x = \sum_{k \in A'} \lambda_k \hat{x}^k + \sum_{k \notin A'} x_k \varepsilon^k, \text{ where } \sum_{k \in A'} \lambda_k = 1, \ \lambda_k \ge 0, \ \forall k \in A'.$$

Then it follows that for every $p \in B_\delta(p^*) \cap \Delta$ and for every $x \in \mathcal{B}_i(p)$,

$$\begin{aligned} p^* \cdot x &= \sum_{k \in A'} \lambda_k p^* \cdot x^k + \sum_{k \notin A'} x_k p^* \cdot \epsilon^k \\ &= \sum_{k \in A} \lambda_k \left(p_k^* \frac{p \cdot e_i}{p_k} \right) \\ &< \sum_{k \in A} \lambda_k (p^* \cdot e_i + \varepsilon) \\ &= p^* \cdot e_i + \varepsilon. \end{aligned}$$

Hence, $x \in V$, *i.e.*, $\mathcal{B}_i(p) \subset V \subset W$ and we conclude that $\mathcal{B}_i$ is upper hemi-continuous. □

(6) Let V be an open subset of R^ℓ_+ such that $\mathcal{B}_i(p) \cap V \neq \emptyset$. That is, there exists $x \in V$ such that $p \cdot x \leq p \cdot e_i$. Consider $x^\alpha = \alpha \hat{x} + (1-\alpha)x \in X$ with $\alpha \in (0,1)$. For sufficiently small $\alpha > 0$, we have $p \cdot x^\alpha < p \cdot e_i$ and $x^\alpha \in V$. Then there exists $\delta > 0$ such that for every $p' \in B_\delta(p) \cap \Delta$, $p' \cdot x^\alpha < p' \cdot e_i$, *i.e.*, $x^\alpha \in \mathcal{B}_i(p')$. Hence $\mathcal{B}_i(p') \cap V \neq 0$ and we conclude that $\mathcal{B}_i$ is lower hemi-continuous. □

N.B Let $\mathcal{B}_i(p) \subset Y$ for all $p \in \Delta$ with Y is a compact convex subset of X. Then $\mathcal{B}_i$ is compact-valued. Here we can prove the upper hemi-continuity of $\mathcal{B}_i$ by just showing that it has closed graph. For it is upper hemi-continuous iff it has closed graph. For the lower hemi-continuity of $\mathcal{B}_i$, we may use the sequential characterization.

2.

(1) Clearly it is nonempty. By Theorem 1.10 (4) of the handout, it is compact. By Theorem 2.4 (2), it is convex. □

(2) By Maximum Theorem, φ_i is nonempty-valued, compact-valued, and upper hemi-continuous. Futhermore it is convex-valued since u_i is quasi-concave. To see this, let $x, x' \in \varphi_i(p)$. Then, by quasi-concavity of u_i, $u_i(\alpha x + (1-\alpha)x') \geq \min\{u_i(x), u_i(x')\}$ for $\alpha \in [0,1]$. Since $\alpha x + (1-\alpha)x' \in \mathcal{B}_i(p)$ by convexity of $\mathcal{B}_i(p)$ and $u_i(x) = u_i(x')$ is a maximum of u_i on $\mathcal{B}_i(p)$, $u_i(\alpha x + (1-\alpha)x')$ is also maximum on $\mathcal{B}_i(p)$, *i.e.*, $\alpha x + (1-\alpha)x' \in \varphi_i(p)$. Hence it is convex-valued. □

(3) Note that $K = C + \{-\sum_i e_i\}$. Since C and $\{-\sum_i e_i\}$ are nonempty, compact, and convex, K is nonempty, compact, and convex. For every i, φ_i and $\{-e_i\}$ are nonempty-valued, compact-valued, convex-valued, and upper hemi-continuous, $\varphi_i(\cdot) + \{-e_i\}$ is nonempty-valued, compact-valued, convex-valued, and upper hemi-continuous [Theorem 2.4 (2) and Theorem 3.11 (1)]. By Theorem 2.4 (2) and Theo-

rem 3.11 (1) again, we conclude that Ψ is nonempty-valued, compact-valued, convex-valued, and upper hemi-continuous. $\square$

(4) Fix $p \in \Delta$. Let $z \in \Psi(p)$. Then there is $x_i \in \varphi_i(p)$ such that $z = \sum_{i \in I}(x_i - e_i)$. Since $x_i \in \varphi_i(p)$, $x_i \in \mathcal{B}_i(p)$, *i.e.*, $p \cdot x_i \leq p \cdot e_i$. Thus, $p \cdot (x_i - e_i) \leq 0$ for every $i \in I$. Summing up over i, we obtain $p \cdot z = \sum_{i \in I} p \cdot (x_i - e_i) \leq 0$. $\square$

(5) Let $\Gamma : K \to 2^{R^\ell_+}$ be a constant correspondence such that $\Gamma(z) = \Delta$. Then the correspondence Γ is nonempty-valued, compact-valued, convex-valued, and continuous. Note that $(\cdot) \cdot z$ is a continuous function and that

$$\mu(z) = \{p \in \Delta : p \cdot z = \max_{p' \in \Gamma(z)} p' \cdot z\}.$$

By Maximum Theorem, μ is nonempty-valued, compact-valued, and upper hemi-continuous. Moreover, since $(\cdot) \cdot z$ is quasi-concave, μ is convex-valued. $\square$

(6) Define the correspondence $\mu \times \Psi : \Delta \times K \to 2^{\Delta \times K}$ by $(\mu \times \Psi)(p, z) = \mu(z) \times \Psi(p)$. Then by Theorem 2.4 (2) and Theorem 3.10 (2), $\mu \times \Psi$ is nonempty-valued, compact-valued, convex-valued, and upper hemi-continuous on the nonempty compact convex set $\Delta \times K$. By the Kakutani fixed point theorem, there is a fixed point $(p^*, z^*) \in \mu(z^*) \times \Psi(p^*)$. This implies that $p^* \cdot z^* \geq p \cdot z^*$ for all $p \in \Delta$ and $p^* \cdot z^* \leq 0$ by (4). Therefore, $p \cdot z^* \leq 0$ for all $p \in \Delta$ so that $z^* \leq 0$. Hence $z^* \in \Psi(p^*) \cap R^\ell_-$. $\square$

3.

(1) $P^{-1}(y) = \{x \in R^\ell_+ : y \in P(x)\} = \{x \in R^\ell_+ : u(y) > u(x)\}$. Since u is continuous, $P^{-1}(y) = u^{-1}((-\infty, u(y)))$ is open in R^ℓ_+. $\square$

(2) For every open[18] set V in R^ℓ_+, the following holds

$$\bigcup_{y \in V} P^{-1}(y) = P^-(V) := \{x \in R^\ell_+ : P(x) \cap V \neq \emptyset\}.$$

Since the left hand side is the union of open sets in X, it is also open in X. This implies that $P^-(V)$ is open and P is lower hemi-continuous. $\square$

4. Choose any open subset W of Y. Let $S = \{x \in X : \psi(x) \subset W\}$. Then $S = A \cup B$ where $A = \{x \in V : f(x) \in W\} = f^{-1}(W)$ and $B = \{x \in X : \varphi(x) \subset W\} = \varphi^+(W)$, because f is a continuous selection of φ on V. Since f is continuous, A is open in X. Since φ is upper hemi-continuous, $\varphi^+(W)$ is open in X. Because the union of two open sets is open, we conclude that ψ is upper hemi-continuous. $\square$

5.

[18] The openess of V is not necessary for the fact.

(1) Choose $(x^*, y^*) \in Gr(coP)$. Then, $y^* \in coP(x^*)$, so that $y^* = \sum_i \alpha_i y_i$ with $\alpha \in \Delta$ and $y_i \in P(x^*)$. Because P has an open graph, there are open sets U_i's and V_i's such that $(x^*, y_i) \in U_i \times V_i \subset Gr(P)$. Let $U = \bigcap_i U_i$ and $V = \sum_i \alpha_i V_i$. Then U and V are open and $x^* \in U$ and $y^* \in V$. To show $U \times V \subset Gr(coP)$, take $(x, y) \in U \times V$. Then $y = \sum_i \alpha_i y_i'$ with $(x, y_i') \in U_i \times V_i \subset Gr(P)$. That is, $y_i' \in P(x)$ for each i. It follows that $y \in coP(x)$, so that $(x, y) \in Gr(coP)$, *i.e.*, $U \times V \subset Gr(coP)$. Hence, coP has an open graph. □

(2) Consider any open subset V of Y. Let $S = \{x \in X : \mu(x) \cap V \neq \emptyset\}$. To show that S is open, consider $x^* \in S$. By definition, there is $y^* \in coP(x^*) \cap A(x^*) \cap V$. Because coP has an open graph, there are open sets U and W such that $(x^*, y^*) \in U \times W \subset Gr(coP)$. Let $B = V \cap W$, which is open. Let $C = \{x \in X : A(x) \cap B \neq \emptyset\}$, which is open because A is lower hemi-continuous. Note that $x^* \in C \cap U$, which is open and $y^* \in \mu(x^*) \cap B$. To show that $C \cap U \subset S$, choose any $x \in C \cap U$. Since $x \in C$, there is $y \in A(x) \cap B$. Therefore, $(x, y) \in U \times V \subset Gr(coP)$, *i.e.*, $y \in coP(x)$. It follows that $y \in \mu(x) \cap V$. Hence $x \in S$. □

HOMEWORK 2

1. A **relation** $\mathcal{R}$ is a correspondence from X to 2^X. We have definitions on $\mathcal{R}$ as follows.

- $\mathcal{R}$ is **reflexive** if, for every $x \in X$, $x \in \mathcal{R}(x)$.
- $\mathcal{R}$ is **irreflexive** if, for every $x \in X$, $x \notin \mathcal{R}(x)$.
- $\mathcal{R}$ is **complete** if, for every $x, x' \in X$, $x' \in \mathcal{R}(x)$ or $x \in \mathcal{R}(x')$.
- $\mathcal{R}$ is **transitive** if $x'' \in \mathcal{R}(x')$ and $x' \in \mathcal{R}(x)$ implies $x'' \in \mathcal{R}(x)$.
- $\mathcal{R}$ is **negatively transitive** if $x'' \notin \mathcal{R}(x')$ and $x' \notin \mathcal{R}(x)$ implies $x'' \notin \mathcal{R}(x)$.
- $\mathcal{R}$ is **symmetric** if $x' \in \mathcal{R}(x)$ implies $x \in \mathcal{R}(x')$.
- $\mathcal{R}$ is **asymmetric** if $x' \in \mathcal{R}(x)$ implies $x \notin \mathcal{R}(x')$.
- $\mathcal{R}$ is **antisymmetric** if $x' \in \mathcal{R}(x)$ and $x \in \mathcal{R}(x')$ implies $x' = x$.

(1) Let $R : X \to 2^X$ be a relation on X. If R is complete, then is it reflexive too?

(2) Let $P : X \to 2^X$ be a relation on X. Show the following.

  (a) If P is asymmetric, then it is irreflexive.
  (b) If P is asymmetric and negatively transitive, then it is transitive.

(3) Define $R(x) := \{x' \in X : x \notin P(x')\}$ and $I(x) = \{x' \in X : x' \in R(x) \text{ and } x \in R(x')\}$. Show that

  (a) P is asymmetric iff R is complete.
  (b) P is negatively transitive iff R is transitive.
  (c) P is asymmetric and negatively transitive implies that I is reflexive, symmetric, and transitive.

N. B. Note that $R(x) = X \setminus P^{-1}(x)$ and $I(x) = R(x) \cap R^{-1}(x)$. A binary relation $\succ$ on X defines the correspondence $P : X \to 2^X$ by $P(x) = \{x' \in X : x' \succ x\}$ for every $x \in X$. Define $\succeq$ such that $x' \succeq x$ iff $x \not\succ x'$ and define $\sim$ such that $x' \sim x$ iff $x' \succeq x$ and $x \succeq x'$. Then $R(x) = \{x' \in X : x' \succeq x\}$ and $I(x) = \{x' \in X : x' \sim x\}$.

2. Let $\succeq$ be a preference on $X = R^\ell_+$ and define $\succ$ such that $x' \succ x$ iff $x \not\succeq x'$. Define correspondences $R : X \to 2^X$ and $P : X \to 2^X$ by $R(x) = \{x' \in X : x' \succeq x\}$ and $P(x) = \{x' \in X : x' \succ x\}$, respectively.

- $\succeq$ is **reflexive (complete, transitive)** if R is reflexive (complete, transitive).
- $\succeq$ is **continuous** if R has closed sections, *i.e.*, $R(x)$ and $R^{-1}(x)$ are closed in X for every $x \in X$.
- $\succeq$ is **weakly monotone** if $x' \geq x$ with $x, x' \in X$ implies $x' \in R(x)$.
- $\succeq$ is **monotone** if $x' \gg x$ with $x, x' \in X$ implies $x' \in P(x)$.
- $\succeq$ is **strongly monotone** if $x' > x$ with $x, x' \in X$ implies $x' \in P(x)$.
- $\succeq$ is **nonsatiated** if P is nonempty-valued.
- $\succeq$ is **locally nonsatiated** if, for every $x \in X$ and for every $\varepsilon > 0$, $B_\varepsilon(x) \cap P(x) \neq \emptyset$.
- $\succeq$ is **convex** if R is convex-valued.
- $\succeq$ is **strictly convex** if for every $x \in X$, $x', x'' \in R(x)$ and $x' \neq x''$ imply $\alpha x' + (1-\alpha)x'' \in P(x)$ for every $\alpha \in (0, 1)$.

(1) Show that if $\succeq$ is complete and transitive on X, then two different indifference sets cannot intersect. Recall that an indifference set at x is defined by $I(x) = \{x' \in X : x' \sim x\}$.

(2) Show that if $\succeq$ is complete and transitive on X, then $x'' \in P(x')$ and $x' \in R(x)$ imply $x'' \in P(x)$.

(3) Show the following.

(a) If $\succeq$ is strongly monotone, then it is monotone.

(b) If $\succeq$ is monotone, then it is locally nonsatiated.

(c) If $\succeq$ is transitive, locally nonsatiated, and weakly monotone, then it is monotone.

(d) If $\succeq$ is locally nonsatiated, then it is nonsatiated.

(e) Let $\succeq$ be complete and transitive. Then $\succeq$ is convex iff $\succ$ is convex, *i.e.*, the correspondence P is convex-valued.

(4) Give an example illustrating a convex preference that is locally nonsatiated but is not monotone.

(5) Show that $\succeq$ is continuous iff the correspondence P has open sections, *i.e.*, $P(x)$ and $P^{-1}(x)$ are open in X for every $x \in X$.

(6) Show that $\succeq$ is continuous iff the correspondence R has closed graph, *i.e.*, $x_n \to x$, $y_n \to y$, and $y_n \in R(x_n)$ for every n imply $y \in R(x)$.

3. Let $\succeq$ be complete and transitive. We can define the convexities of preference $\succeq$ in the following forms.

(a) $x' \succeq x$ implies $\alpha x' + (1-\alpha)x \succeq x$ for every $\alpha \in [0,1]$.

(b) $x' \succ x$ implies $\alpha x' + (1-\alpha)x \succ x$ for every $\alpha \in (0,1]$.

(c) $x' \sim x$ with $x' \neq x$ implies $\alpha x' + (1-\alpha)x \succ x$ for every $\alpha \in (0,1)$.

(1) Let $\succeq$ be continuous. Show that (c) implies (b), which, in turn, implies (a).

(2) Show that (a) holds iff $\succeq$ is convex for every $x \in X$.

(3) Show that if $\succeq$ is convex, continuous, strongly monotone, then (b) holds.

(4) When is it true that (c) holds iff $\succeq$ is strictly convex?

(5) Show that if $\succeq$ is nonsatiated and (b) holds, then $\succeq$ is locally nonsatiated.

SOLUTION 2

1.

(1) Yes. Since R is complete, $x \in R(x)$ for every $x \in X$.

(2) (a) Suppose not. Then there is $x \in P(x)$. By asymmetry, $x \notin P(x)$, a contradiction. $\square$

(b) Suppose not. Then there is x, x', and x'' such that $x'' \in P(x')$ and $x' \in P(x)$ but $x'' \notin P(x)$. By asymmetry, we have $x \notin P(x')$. It follows from negative transitivity that $x'' \notin P(x')$, a contradiction. $\square$

(3) (a) ($\Rightarrow$) If R is not complete, there are x and x' such that $x \notin R(x')$ and $x' \notin R(x)$. By definition of R, we have $x' \in P(x)$ and $x \in P(x')$, which is a contradiction to the asymmetry of P. ($\Leftarrow$) If P is not asymmetric, there are x and x' such that $x \in P(x')$ and $x' \in P(x)$. Then $x' \notin R(x)$ and $x \notin R(x')$, a contraction to the completeness of R. $\square$

(b) ($\Rightarrow$) If R is not transitive, there are x, x', and x'' such that $x'' \in R(x')$ and $x' \in R(x)$ but $x'' \notin R(x)$. Therefore, $x' \notin P(x'')$, $x \notin P(x')$, and $x \in P(x'')$. This is a contradiction to the negative transitivity of P.

($\Leftarrow$) If P is not negatively transitive, there are x, x', and x'' such that $x'' \notin P(x')$ and $x' \notin P(x)$ but $x'' \in P(x)$. Then, by definition, $x' \in R(x'')$ and $x \in R(x')$ but $x \notin R(x'')$, a contradiction to transitivity of R. $\square$

(c) If I is not reflexive, there is x such that $x \notin I(x)$. Then, by definition, $x \notin R(x)$, a contradition. On the other hand, if I is not symmetric, there are x and x' such that $x \in I(x')$ but $x' \notin I(x)$. Then, by definition, $x \in R(x')$ and $x' \in R(x)$, but $x' \notin R(x)$ or $x \notin R(x')$. This is a contradiction. Finally, if I is not transitive, there are x, x', and x'' such that $x'' \in I(x')$ and $x' \in I(x)$ but $x'' \notin I(x)$. Then $x'' \in R(x')$, $x' \in R(x'')$, $x' \in R(x)$, and $x \in R(x')$, but $x'' \notin R(x)$ or $x \notin R(x'')$. This is a contradiction to transitivity of R. $\square$

2.

(1) A indifference set $I(x)$ is defined by

$$I(x) = \{x' \in X : x' \sim x\} = R(x) \cap R^{-1}(x).$$

Note that reflexivity guarantees that $I(x) \neq \emptyset$. Suppose that two different indifference sets intersect on a set A which contains $x \in X$. Then we can write

$x \in A = I_1(x) \cap I_2(x)$. Since $I_1(x)$ and $I_2(x)$ are different, there exist x_1 and x_2 such that $x_i \in I_i(x) \setminus A, i = 1, 2$ and $x_1 \not\sim x_2$. By completeness of preference, we can assume that $x_1 \succ x_2$. It follows that $x_1 \succ x_2 \sim x$. By transitivity of preference, $x_1 \succ x$. This contradicts that $x_1 \in I_1(x)$. Hence, these two indifference set cannot intersect. $\square$

(2) Suppose not. Then $x'' \notin P(x)$. Then $x \in R(x'')$. By transitivity of $\succeq$, we have $x' \in R(x'')$. Therefore, $x'' \notin R(x')$, a contradiction. $\square$

(3) (a) Assume that $\succeq$ is strongly monotone. Let $x, x' \in X$ satisfy $x' \gg x$. Then $x' > x$. Since $\succeq$ is strongly monotone, $x' \in P(x)$. Hence $\succeq$ is monotone. $\square$

(b) Assume that $\succeq$ is monotone. For every $x \in X$ and every $\varepsilon > 0$, choose $x' = x + (\sqrt{\varepsilon/\ell})(1, 1, \ldots, 1)$. Then ¿$x' \gg x$ and $x' \in B_\varepsilon(x)$. Moreover, by monotonicity of $\succeq$, $x' \in P(x)$. Hence $\succeq$ is locally nonsatiated. $\square$

(c) Suppose not. Then there are x, x' in X such that $x' \gg x$ and $x' \notin P(x)$. This means $x \succeq x'$. Since $\succeq$ is locally nonsatiated, there is $\varepsilon > 0$ and x'' such that $x' \gg x''$ and $x'' \in B_\varepsilon(x) \cap P(x)$. By weak monotonicity of $\succeq$, we have $x' \succeq x''$. Moreover $x'' \succ x$. By transitivity of $\succeq$, $x' \succ x$, a contradiction. $\square$

(d) Assume that $\succeq$ is locally nonsatiated. Then since for every $x \in X$ and every $\varepsilon > 0$, $\emptyset \neq B_\varepsilon(x) \cap P(x) \subset P(x)$. Hence, $\succeq$ is nonsatiated. $\square$

(e) ($\Rightarrow$) Let $x' \succ x$, $x'' \succ x$, and $x^\alpha = \alpha x' + (1 - \alpha)x''$. By completeness of $\succeq$, we can assume $x' \succeq x''$. The convexity of $R(x)$ implies that $x^\alpha \succeq x''$. By transitivity of $\succeq$, we have $x^\alpha \succ x$. ($\Leftarrow$) Suppose not. Then there are x', x'', and $x^\alpha = \alpha x' + (1 - \alpha)x''$ such that $x' \succeq x$, $x'' \succeq x$, but $x \succ x^\alpha$. From the transitivity of $\succeq$, $x' \succ x^\alpha$ and $x'' \succ x^\alpha$. The convexity of $P(x)$ implies $x^\alpha \succ x^\alpha$, a contradiction. $\square$

(4) Let $X = R^2_+$. Consider a preference $\succeq$ which is represented by the utility function $u(x, y) = y - x^2$. It is clear that $\succeq$ is convex and locally nonsatiated. Choose $(x^*, y^*) = (1, 1)$ and $(x', y') = (2, 2)$. Then $(x', y') \gg (x^*, y^*)$ but $u(x^*, y^*) > u(x', y')$. Hence $\succeq$ is not monotone.

(5) Note that $R(x) = X \setminus P^{-1}(x)$ and $R^{-1}(x) = X \setminus P(x)$. Hence, $\succeq$ is continuous $\Leftrightarrow$ $R(x)$ and $R^{-1}(x)$ are closed in X for every $x \in X$ $\Leftrightarrow$ $P(x)$ and $P^{-1}(x)$ are open in X for every $x \in X$. $\square$

(6) We should assume that $\succeq$ is complete and transitive. ($\Leftarrow$) Since R has closed graph, it has closed sections. : Choose any $x \in X$. Take any sequence $\{y_n\}$ in $R(x)$ converging

to y. Since $x \to x$, $y_n \to y$, and $y_n \in R(x)$, $y \in R(x)$. This measns that $R(x)$ is closed. In the same way, $R^{-1}(x)$ is closed. Hence $\succeq$ is continuous. ($\Rightarrow$) Let $x_n \to x$, $y_n \to y$, and $y_n \in R(x_n)$ for every n. Suppose that $y \notin R(x)$. Since $R(x)$ and $R^{-1}(y)$ are closed due to the continuity of $\succeq$ and $R(x) \cap R^{-1}(y) = \emptyset$, there is $x^* \in X$ such that $x^* \notin R(x)$ and $x^* \notin R^{-1}(y)$, *i.e.*, $x \succ x^* \succ y$. Since $\succeq$ is continuous ($P(x^*)$ is open) and $x_n \to x$, for sufficiently large n, $x_n \in P(x^*)$. Since $\succeq$ is continuous ($P^{-1}(x^*)$ is open) and $y_n \to y$, for sufficiently large n, $y_n \in P^{-1}(x^*)$. By transitivity of $\succeq$, $x_n \in P(y_n)$ for sufficiently large n. This implies that $y_n \notin R(x_n)$, a contradiction $\square$

3.

(1) [$(c) \Rightarrow (b)$] Suppose not. Then there is a $\alpha \in (0,1]$ and x, x' such that $x' \succ x$ but $x \succeq x^\alpha = \alpha x' + (1-\alpha)x$. If $x \succ x^\alpha$, by continuity, we can find $x^\beta = \beta x + (1-\beta)x^\alpha$ ($\beta \in (0,1)$ is close to 1) such that $x' \succ x^\beta \succ x^\alpha$. If $x \sim x^\alpha$, by (c) and continuity, we can find x^β such that $x' \succ x^\beta \succ x^\alpha$. Therefore, we have $x' \succ x^\beta \succ x^\alpha$. By continuity, we can find [19] $x^\lambda = \lambda x' + (1-\lambda)x^\alpha$ with $\lambda \in (0,1)$ such that $x^\lambda \sim x^\beta$. Then (c) implies that $x^\alpha = \delta x^\lambda + (1-\delta)x^\beta \succ x^\beta$ where $\delta = \alpha\beta/[\alpha\beta + (1-\alpha)\lambda]$, a contradiction. $\square$ [$(b) \Rightarrow (a)$] Suppose not. Then there are x, x' and $\alpha \in (0,1]$ such that $x' \succeq x$ but $x \succ x^\alpha = \alpha x' + (1-\alpha)x$. Note that $x' \succ x^\alpha$ by transitivity. The continuity implies that there is $\beta \in [0,1]$ close to α such that $x \succ x^\beta = \beta x' + (1-\beta)x$ and x^β lies on the open line segment (x, x^α) with $\beta > \alpha$. Then $x^\beta = \lambda x' + (1-\lambda)x^\alpha$ with $\lambda = (\beta-\alpha)/(1-\alpha)$ and $x^\alpha = \delta x + (1-\delta)x^\beta$ with $\delta = (\beta-\alpha)/\beta$. By (b), $x' \succ x^\alpha$ implies $x^\beta \succ x^\alpha$, and $x \succ x^\beta$ implies $x^\alpha \succ x^\beta$, a contradiction. $\square$

(2) ($\Rightarrow$) Let $x' \succeq x$, $x'' \succeq x$, and $x^\alpha = \alpha x' + (1-\alpha)x''$. By completeness of $\succeq$, we can assume that $x' \succeq x''$. It follows from (a) that $x^\alpha \succeq x''$. By transitivity of $\succeq$, it follows that $x^\alpha \succeq x$, *i.e.*, $x^\alpha \in R(x)$. Hence we conclude that $\succeq$ is convex.

($\Leftarrow$) Let $x' \succeq x$ and $x^\alpha = \alpha x' + (1-\alpha)x$. Because of completeness of $\succeq$, $x \succeq x$. Because $R(x)$ is convex, $x^\alpha \succeq x$. $\square$

(3) Suppose not. Then there are x, x' and $x^\alpha = \alpha x' + (1-\alpha)x$ with $\alpha \in (0,1]$ such that $x' \succ x \succeq x^\alpha \in$. Note that since $\succeq$ is strongly monotone, $x' \in R^\ell_+ \setminus \{0\}$. Without loss of generality, we can assume $x'_1 > 0$. By continuity of $\succeq$, we can find

[19] Define $\lambda = \sup\{t \in [0,1] : x^\beta \succeq tx' + (1-t)x^\alpha\}$. Now suppose $x^\lambda \succ x^\beta \succ x^\alpha$. By continuity, for some $\theta \in (0,1)$ close to 1, $x^\theta = \theta x^\lambda + (1-\theta)x^\alpha \succ x^\beta$. But since $x^\theta = \lambda\theta x' + (1-\lambda\theta)x^\alpha$ with $\lambda\theta \in [0,\lambda)$, the definition of λ implies $x^\beta \succeq x^\theta$, a contradiction. Suppose $x' \succ x^\beta \succ x^\lambda$. By continuity, for some $\theta \in (0,1)$ close to 1, $x^\beta \succ x^\theta = \theta x^\lambda + (1-\theta)x'$. But since $x^\theta = (1-\theta(1-\lambda))x' + \theta(1-\lambda)x^\alpha$ with $1-\theta(1-\lambda) > \lambda$, the definition of λ implies that $x^\theta \succ x^\beta$, a contradiction. Hence $x^\lambda \sim x^\beta$.

$\epsilon = (\varepsilon, 0, 0, \ldots, 0)$ with $\varepsilon > 0$ such that $x' - \epsilon \succ x$. By completeness of $\succeq$, $x \succeq x$. By convexity of $\succeq$, we have $x^\alpha - \alpha\epsilon \succeq x$, a contradiction to strong monotonicity. $\square$

(4) If $\succeq$ is continuous, then both are equivalent.

($\Rightarrow$) Suppose not. Then there are x, x', x'', and x^α with $x' \neq x''$ such that $x' \succeq x$ and $x'' \succeq x$, but $x \succeq x^\alpha = \alpha x' + (1-\alpha)x''$. There are three case to consider.

(*i*) Consider the case where $x \succ x^\alpha$. Because $x' \succeq x \succ x^\alpha$, by continuity, we can find $x^\beta = \beta x' + (1-\beta)x^\alpha \sim x$ with $\beta \in (0, 1]$. Similarly, we can find $x^\lambda = \lambda x'' + (1-\lambda)x^\alpha \sim x$ with $\lambda \in (0, 1]$. By transitivity, $x^\lambda \sim x^\beta$. Note that $x^\lambda \neq x^\beta$. However, $x^\alpha = \delta x^\lambda + (1-\delta)x^\beta$ with $\delta \in (0, 1)$ so that (*c*) implies $x^\alpha \succ x^\beta \sim x$, a contradiction.

(*ii*) Consider the case of $x' \succ x \sim x^\alpha$. By completeness of $\succeq$, we can assume $x'' \succeq x'$ without loss of generality. Note that (*c*) imlies (*b*). By (*b*) and continuity, we can find $x^\beta = \beta x' + (1-\beta)x^\alpha$ with $\beta \in (0, 1)$ such that $x' \succ x^\beta \succ x^\alpha$. Because $x'' \succ x^\beta \succ x^\alpha$, by continuity we can find $x^\lambda = \lambda x'' + (1-\lambda)x^\alpha$ with $\lambda \in (0, 1)$ such that $x^\lambda \sim x^\beta$. This with $x^\lambda \neq x^\beta$ implies $x^\alpha = \delta x^\lambda + (1-\delta)x^\beta \succ x^\beta$ with $\delta \in (0, 1)$, a contradiction.

(*iii*) Finally, consider the case $x' \sim x \sim x^\alpha$. By completeness of $\succeq$, we may assume $x'' \succeq x'$. If $x'' \sim x'$, (*c*) implies $x^\alpha \succ x'$, a contradiction. If $x'' \succ x'$, recall that (*c*) with continuity implies (*b*) to conclude that $x^\alpha \succ x'$, a contradiction.

($\Leftarrow$) Let $x' \sim x$ with $x' \neq x$. Then $x' \in R(x)$ and $x \in R(x)$. By the definition of strict convexity of $\succeq$, we obtain that $\alpha x' + (1-\alpha)x \in P(x)$ for $\alpha \in (0, 1)$. $\square$

(5) Assume that $\succeq$ is nonsatiated and (b) holds. Take any $x \in X$ and any $\varepsilon > 0$. Since $\succeq$ is nonsatiated, there is $x' \in P(x)$. By (b), $x^\alpha = \alpha x' + (1-\alpha)x \in P(x)$ for every $\alpha \in (0, 1]$. However, for sufficiently small $\alpha > 0$, $x^\alpha \in B_\varepsilon(x)$. It follows that $x^\alpha \in B_\varepsilon(x) \cap P(x)$, *i.e.*, $\succeq$ is locally nonsatiated. $\square$

HOMEWORK 3

1. Let $X = R_+^\ell$ be the consumption set of a consumer, $e \in X$ be his endowment, and $\succ$ be his preference relation. Assume that $\succ$ is irreflexive, continuous, and convex.[20] Show that, for all $p \gg 0$, there is always a best element in the budget set $\mathcal{B}(p)$ of our consumer, *i.e.,* there always exists a $x \in \mathcal{B}(p)$ such that there is no $x' \in \mathcal{B}(p)$ with $x' \succ x$. (Hint : Use the Michael Selection Theorem.)

2. Suppose that preference $\succeq$ is complete, transitive, and locally nonsatiated. Let $\mathcal{B}(p) = \{x \in R_+^\ell : p \cdot x \leq p \cdot e\}$. Let $\varphi : \Delta \to 2^{R_+^\ell}$ the demand correspondence of the consumer with endowment $e \in X = R_+^\ell$, *i.e.* $x^* \in \varphi(p)$ implies that $x^* \in \mathcal{B}(p)$ and $x^* \succeq x$ for every $x \in \mathcal{B}(p)$.

(1) Show that $p \cdot x = p \cdot e$ for every $x \in \varphi(p)$.

(2) Let $x^* \in \varphi(p)$. If $x \succ x^*$, then $p \cdot x \geq p \cdot x^*$.

(3) If the preference $\succeq$ is convex, then φ is convex-valued, *i.e.*, $\varphi(p)$ is convex.

(4) If the preference $\succeq$ is strictly convex, then φ is singleton-valued, *i.e.*, φ becomes demand function.

3. Consider one individual with consumption set $X = R_+^\ell$ and preference $\succeq$.

(1) State conditions on the preference to ensure that there exists $p \in R^\ell \setminus \{0\}$ such that $x' \succeq x$ implies $p \cdot x' \geq p \cdot x$.

(2) Verify (1).

(3) What kind of assumption on the preference guarantees that $p \in R_+^\ell$?

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[20] Define a correspondence $P : X \to 2^X$ by $P(x) = \{x' \in X : x' \succ x\}$.

(1) $\succ$ is **irreflexive** if $x \notin P(x)$.

(2) $\succ$ is **transitive** if $x \in P(y)$ and $y \in P(z)$ implies $x \in P(z)$.

(3) $\succ$ is **continuous** if $P(x)$ and $P^{-1}(x)$ are open in X for every $x \in X$.

(4) $\succ$ is **weakly monotone** if $x' \geq x$ implies $x \notin P(x')$.

(5) $\succ$ is **monotone** if $x' \gg x$ implies $x' \in P(x)$.

(6) $\succ$ is **strongly monotone** if $x' > x$ implies $x' \in P(x)$.

(7) $\succ$ is **convex** if P is convex-valued.

(8) $\succ$ is **strictly convex** if $x \notin P(x') \cap P(x'')$ and $x' \neq x''$ imply that $\alpha x' + (1-\alpha)x'' \in P(x)$ for every $\alpha \in (0,1)$.

4. Consider one individual with consumption set $X = R^\ell_+$, endowment $e \in intX$ and preference $\succ$. Assume that $\succ$ is irreflexive, transitive, continuous, strongly monotone, and strictly convex. Define a correspondence $\varphi : X \to 2^X$ by

$$\varphi(p) := \{x \in X : p \cdot x \leq p \cdot e \text{ and } x' \succ x \text{ implies } p \cdot x' > p \cdot x\}.$$

(1) Show that φ is singleton-valued.

(2) Show that $P(e) := \{x \in X : x \succ e\}$ has nonempty interior and $e \notin P(e)$.

(3) Show that, for some $p^* \in \Delta$, $x \succ e$ implies that $p^* \cdot x \geq p^* \cdot e$.

(4) Show that there exists $\hat{x} \in X$ such that $\hat{x} \ll e$ and that, for the $p^* \in \Delta$ in (3), $x \succ e$ implies that $p^* \cdot x > p^* \cdot e$. Conclude that there is $p^* \in \Delta$ such that $\varphi(p^*) = \{e\}$.

5. Consider a two-goods two-agents economy where each agent has the following utility function and endowment. Let $X = R^2_+$. Find equilibrium prices and allocations.

(1)	$u_1(x, y) = x + y + xy$	$e_1 = (1, 0)$	(2)	$u_1(x, y) = \min\{2x, y\}$	$e_1 = (2, 8)$
	$u_2(x, y) = xy$	$e_2 = (0, 1)$		$u_2(x, y) = \min\{x, 3y\}$	$e_2 = (6, 0)$
(3)	$u_1(x, y) = 2x + y,$	$e_1 = (0, 1)$	(4)	$u_1(x, y) = \log x + y$	$e_1 = (1, 0)$
	$u_2(x, y) = xy,$	$e_2 = (1, 0)$		$u_2(x, y) = \log x + 2y$	$e_2 = (0, 1).$

SOLUTION 3

1. It follows from 3 that $p \gg 0$ impiles that $B(p)$ is nonempty compact convex subset of R^ℓ_+. Let us define a correspondence $P : B(p) \to 2^{B(p)}$ by $P(x) = \{x' \in B(p) : x' \succ x\}$. By continuity of preference, P has open lower sections so that it is lower hemi-continuous. By convexity of preference, P is convex-valued. Now suppose that there is no best element in $B(p)$. Then for every $x \in B(p)$, $P(x) \neq \emptyset$, *i.e.*, P is nonempty-valued. By Michael Selection Theorem,[21] there is a continuous selection $f : B(p) \to B(p)$ of P. Brower Fixed Point Theorem implies that there exists $x^* \in B(p)$ such that $x^* = f(x^*) \in P(x^*)$. This contradicts the irreflexivity of preference. Hence, there is a best element in $B(p)$. $\square$

2.

(1) Suppose not. Then there is $x^* \in S = \{x \in R^\ell_+ : p \cdot x < p \cdot e\}$. Since $f(x) = p \cdot x$ is a continuous function from R^ℓ_+ to R, $S = f^{-1}((-\infty, p \cdot e))$ is open in R^ℓ_+. Thus there exists an $\varepsilon > 0$ such that $B_\varepsilon(x^*) \cap R^\ell_+ \subset S$. However, by the local nonsatiation of preferences, there exists $z \in B_\varepsilon(x^*) \cap R^\ell_+$ such that $z \succ x^*$. It follows that $p \cdot z \leq p \cdot e$ and $x \succ x^*$. This contradicts that x^* is a maximizer. Hence, $p \cdot x^* = p \cdot e$. $\square$

(2) Suppose that $x \succ x^*$ but $p \cdot x < p \cdot x^*$. Then $p \cdot x < p \cdot e$ and $x \succ x^*$. This contradicts that $x^* \in \varphi(p)$. Hence, if $x \succ x^*$, then $p \cdot x \geq p \cdot x^*$. $\square$

(3) Let $x', x'' \in \varphi(p)$ and $x^\alpha = \alpha x' + (1-\alpha)x''$. Since $B(p)$ is convex, $x^\alpha \in B(p)$. By convexity of $\succeq$, $x^\alpha \succeq x'$. Since $x' \succeq x$ for every $x \in B(p)$, $x^\alpha \succeq x$ for every $x \in B(p)$. This means that $x^\alpha \in \varphi(p)$, *i.e.*, $\varphi(p)$ is convex. $\square$

(4) Suppose that $\varphi(p)$ is not singleton. Then we have two elements x' and x^* in $\varphi(p)$. Then $x' \succeq x^*$. And $x^* \succeq x^*$ by completeness of preference. By strict convexity of preference, $x^*/2 + x'/2 \succ x^*$. Furthermore, since $p \cdot x^* \leq p \cdot e$ and $p \cdot x' \leq p \cdot e$, it holds that $p \cdot (x^*/2 + x'/2) \leq p \cdot e$. This contradicts that $x^* \in \varphi(p)$. Hence $\varphi(p)$ has the unique element. $\square$

3.

(1) $\succeq$ is complete, transitive, convex, and locally nonsatiated.

(2) Let $P(x) = \{x' \in X : x' \succ x\}$. Since $\succeq$ is complete, transitive, and convex, $P(x)$ is convex.[22] It is nonempty due to local nonsatiation. Moreover, the

[21] We can directly apply Browder Fixed Point Theorem, which is a corollary of Michael Selection Theorem and Brouwer Fixed Point Theorem, to get a fixed point.

[22] See Homework 2: 2-(3)-(e).

completeness of $\succeq$ imples that $x \notin P(x)$.[23] By Separating Hyperplane Theorem, there is $p \in R^\ell \setminus \{0\}$ such that for every $x' \in P(x)$, $p \cdot x' \geq p \cdot x$. Let $x' \sim x$. By local nonsatiation, we can find $x^n \in B_{1/n}(x') \cap R^\ell_+$ such that $x^n \succ x'$. The completeness and transitivity implies $x^n \succ x$.[24] Thus, $p \cdot x^n \geq p \cdot x$. Because $x^n \to x'$, we obtain $p \cdot x' \geq p \cdot x$. □

(3) If preference is weakly monotone, then $p \in R^\ell_+ \setminus \{0\}$. When $\succeq$ is weakly monotone, $x + \varepsilon_m \succeq x$ for every $x \in X$ where ε_m is a vector whose m-th element is 1 and 0 otherwise. (2) implies $p \cdot (x + \varepsilon_m) \geq p \cdot x$. Then, $p_m \geq 0$. This holds for every $m = 1, \ldots, \ell$. Hence, $p \in R^\ell_+ \setminus \{0\}$. □

4.

(1) Suppose not. There are two different elements x and x' in $\varphi(p)$ for some p. Then $p \cdot x \leq p \cdot e$ and $p \cdot x' \leq p \cdot e$. Let $x'' = (1/2)x + (1/2)x'$. Then $p \cdot x'' \leq p \cdot e$. By strict convexity of $\succ$, we have $x'' \succ x$ or $x'' \succ x'$. The property of φ implies that $p \cdot x'' > p \cdot e$, which is a contradiction. □

(2) By strong monotonicity of $\succ$, $e + (1, 0, \ldots, 0) \succ e$, *i.e.*, $e + (1, 0, \ldots, 0) \in P(e)$. However, by continuity of $\succ$, $P(e)$ is open so that $e + (1, 0, \ldots, 0) \in int[P(e)]$. It follows from irreflexivity that $e \notin P(e)$. □

(3) By strict convexity of $\succ$, $P(e)$ is convex. From (2), $int[P(e)] \neq \emptyset$ and $e \notin P(e)$. By Separating Hyperplane Theorem, there exists $p^* \in R^\ell \setminus \{0\}$ such that, for every $x \in P(e)$, $p^* \cdot x \geq p^* \cdot e$. However, the strong monotonicity of $\succ$ implies $e + \varepsilon_m \succ e$ where ε_m is a vector whose m-th element is 1 and others are 0. Then, $p^* \cdot (e + \varepsilon_m) \geq p^* \cdot e$. Then, $p^*_m \geq 0$. This holds for every $m = 1, \ldots, \ell$. Therefore, $p^* \in R^\ell_+ \setminus \{0\}$. Finally, it follows from the zero homogeneity of the inequality, $p^* \cdot x \geq p^* \cdot e$, that we can take $p^*/\|p^*\| \in \Delta$ instead of p^*. Hence, without loss of generality, $p^* \in \Delta$. □

(4) Note that $p^* \cdot \hat{x} < p^* \cdot e$. Let $x \succ e$. Then by (3), $p^* \cdot x \geq p^* \cdot e$ for some $p^* \in \Delta$. Now suppose that $p^* \cdot x = p^* \cdot e$. Then, for sufficiently small $\alpha \in (0, 1)$, $x^\alpha = (1-\alpha)x + \alpha\hat{x} \succ e$ by continuity of $\succ$. But $p^* \cdot x^\alpha = (1-\alpha)p^* \cdot x + \alpha p^* \cdot \hat{x} < p^* \cdot e$. This is a contradiction. Hence, $x \succ e$ implies $p^* \cdot x > p^* \cdot e$. That is, $e \in \varphi(p^*)$. It follows from (1) that $\varphi(p^*) = \{e\}$. □

5. By the homogeneity of demand correspondence, we can set $(p_x, p_y) = (p, 1)$.

[23] See Homework 2: 1-(2)-(a) and 1-(3)-(a).
[24] See Homework 2: 1-(2), (3).

(1) Consider consumer 1. $L = x_1 + y_1 + x_1 y_1 + \lambda(p - px_1 - y_1)$.

$$\begin{array}{ll} \textbf{(FOC)} & 1 + y_1 - \lambda p \leq 0, \quad x_1 \geq 0, \quad x_1(1 + y_1 - \lambda p) = 0 \\ & 1 + x_1 - \lambda \leq 0, \quad\;\; y_1 \geq 0, \quad y_1(1 + x_1 - \lambda) = 0. \end{array}$$

Note that $1 + y_1 - \lambda p \leq 0$ and $y_1 \geq 0$ imply that $p \neq 0$.

(i) $x_1 = y_1 = 0 : 0 = px_1 + y_1 = p \neq 0$, a contradiction .

(ii) $x_1 > 0, y_1 = 0 : px_1 + y_1 = p$ implies $x_1 = 1$. $\lambda = 1/p \geq 2$. Thus, $p \in (0, 1/2]$.

(iii) $x_1 = 0, y_1 > 0 : \lambda = 1$ and $y_1 = p$. Thus, $1 + p - p \leq 0 = 1$, a contradiction.

(iv) $x_1 > 0, y_1 > 0 : 1 + y_1 = p(1 + x_1)$ and $px_1 + y_1 = p$ imply that $x_1 = 1/2p$ and $y_1 = p - (1/2)$ with $p > 1/2$.

In sum,

$$x_1 = \begin{cases} 1 & \text{if } p \in (0, 1/2] \\ 1/2p & \text{if } p > 1/2 \end{cases}$$

Next, consider consumer 2. Clearly, $y_2/x_2 = p$. Then $px_2 + y_2 = 1$ imples $x_2 = 1/2p$ and $y_2 = 1/2$. It follows from the market clearing condition that the Walrasian equilibrium is given by

$$((p_x, p_y), (x_1, y_1), (x_2, y_2)) = ((1, 1), (1/2, 1/2), (1/2, 1/2)).$$

(2) At optimum, it must be the case that

$$2x_1 = y_1, \quad x_2 = 3y_2.$$

Using budget constraints, we get demand functions

$$(x_1, y_1) = \left(\frac{2p + 8}{p + 2}, \frac{4p + 16}{p + 2} \right), \quad (x_1, y_1) = \left(\frac{18p}{3p + 1}, \frac{6p}{3p + 1} \right).$$

Market clearing conditions implies that the equilibrium is given by

$$((p_x, p_y), (x_1, y_1), (x_2, y_2)) = \left(\left(\frac{4}{3}, 1 \right), \left(\frac{16}{5}, \frac{32}{5} \right), \left(\frac{24}{5}, \frac{8}{5} \right) \right).$$

(3) Consider consumer 1. $L = 2x_1 + y_1 + \lambda(1 - px_1 - y_1)$.

$$\begin{array}{ll} \textbf{(FOC)} & 2 - \lambda p \leq 0, \quad x_1 \geq 0, \quad x_1(2 - \lambda p) = 0 \\ & 1 - \lambda \leq 0, \quad\;\; y_1 \geq 0, \quad y_1(1 - \lambda) = 0. \end{array}$$

(i) $x_1 = y_1 = 0 : px_1 + y_1 = 1$ implies $0 = 1$, contradiction .

(ii) $x_1 > 0, y_1 = 0 :$ $\lambda = 2/p$ and $\lambda \geq 1$. Thus, $p \in (0, 2]$. $px_1 + y_1 = 1$ implies $x_1 = 1/p$.

(iii) $x_1 = 0, y_1 > 0 :$ $2 - \lambda p \leq 0$ and $1 - \lambda = 0$. Thus, $p \geq 2$. $px_1 + y_1 = 1$ implies $y_1 = 1$.

(iv) $x_1 > 0, y_1 > 0 :$ $2 - \lambda p = 1 - \lambda = 0$. Thus, $p = 2$. $2x_1 + y_1 = 1$ implies that $x_1 = \alpha$ and $y_1 = 1 - 2\alpha$ with $\alpha \in (0, 1/2)$.

In sum,

$$x_1 = \begin{cases} 1/p & \text{if } p \in (0, 2] \\ (0, 1/2) & \text{if } p = 2 \\ 0 & \text{if } p \geq 2 \end{cases}$$

Next, consider consumer 2. Clearly, $y_2/x_2 = p$. Then $px_2 + y_2 = p$ imples $x_2 = 1/2$ and $y_2 = 1/2p$. It follows from the market clearing condition that the Walrasian equilibrium is given by

$$((p_x, p_y), (x_1, y_1), (x_2, y_2)) = ((2, 1), (1/2, 0), (1/2, 1)).$$

(4) Consider consumer 1. $L = \log x_1 + y_1 + \lambda_1(p - px_1 - y_1)$

$$\textbf{(FOC)} \quad \begin{aligned} &1/x_1 - \lambda_1 p = 0 \\ &1 - \lambda_1 \leq 0, \qquad y_1 \geq 0, \quad y_1(1 - \lambda_1) = 0. \end{aligned}$$

(i) $y_1 = 0 :$ $\lambda_1 \geq 1$. $px_1 + y_1 = p$ implies $x_1 = 1$ and $p \in (0, 1]$.

(ii) $y_1 > 0 :$ $\lambda_1 = 1$. $px_1 + y_1 = p$ implies $x_1 = 1/p$, $y_1 = p - 1$, and $p > 1$.

Therefore,

$$x_1 = \begin{cases} 1 & \text{if } p \in (0, 1] \\ 1/p & \text{if } p > 1 \end{cases}$$

Similarly, for consumer 2, we have

$$\textbf{(FOC)} \quad \begin{aligned} &1/x_2 - \lambda_2 p = 0 \\ &2 - \lambda_2 \leq 0, \qquad y_2 \geq 0, \quad y_2(2 - \lambda_2) = 0. \end{aligned}$$

(i) $y_2 = 0 :$ $\lambda_2 \geq 2$. $px_2 + y_2 = 1$ implies that $x_2 = 1/p$ and $\lambda_2 = 1$. This is a contradiction.

(ii) $y_2 > 0 :$ $\lambda_2 = 2$. $px_2 + y_2 = 1$ implies that $x_2 = 1/(2p)$, $y_2 = 1/2$, and $p > 0$.

Therefore,

$$x_2 = \frac{1}{2p}, \ \forall p > 0.$$

Using market clearing condition, we have the Walrasian equilibrium

$$((p_x, p_y), (x_1, y_1), (x_2, y_2)) = ((3/2, 1)(2/3, 1/2), (1/3, 1/2)).$$

HOMEWORK 4

1. Consider an economy $\mathcal{E} = \{(X_i, u_i, e_i) : i = 1, \dots, m\}$, where $e_i \in X_i = R^\ell_+$ and $u_i : X_i \to R$ is continuous for every i.

(1) Show that $x^* \in \prod_{i=1}^m X_i$ is a Pareto optimal allocation iff for every i, $x_i^* \in X_i$ is a solution to the following problem :

$$\max_{x_i} \left\{ u_i(x_i) : u_j(x_j) \geq u_j(x_j^*), \ \forall j \neq i, \text{ and } \sum_{i=1}^n x_i \leq \sum_{i=1}^n e_i \right\}$$

(2) Assume that u_i is differentiable for every i. Use the Kuhn-Tucker theorem to provide a Lagrange characterization of a Pareto optimal allocation x^*. Assume that $x^* \gg 0$. Interpret the Lagrange multipliers.

2. Consider an economy $\mathcal{E} = \{(X_i, u_i, e_i) : i = 1, \dots, m\}$, where $e_i \in X_i = R^\ell_+$ and $u_i : X_i \to R_+$ is continuous and increasing for every i. Let $A = \{x \in \prod_i X_i : \sum_i x_i \leq \sum_i e_i\}$. Define the utility set $U = \{u \in R^m_+ : u_i(x_i) = u_i, \ x \in A\}$. A social welfare function $W : \prod_i X_i \to R_+$ is defined by

$$W(x_1, \dots, x_m) = \sum_{i=1}^n \lambda_i u_i(x_i),$$

where $\lambda \in R^m_+ \setminus \{0\}$.

(1) Show that if u_i is concave for every i, then W is concave.

(2) Show that if u_i is concave for every i and is strictly concave for some i, then W is strictly concave.

(3) If u_i is quasi-concave for every i, then is W is quasi-concave ?

(4) Show that if x^* maximizes social welfare function on A for some $\lambda \in R^m_+ \setminus \{0\}$, then x^* is weakly Pareto optimal.

(5) Show that if x^* maximizes social welfare function on A for some $\lambda \in R^m_{++}$, then x^* is Pareto optimal.

(6) Show that if u_i is concave for every i and x^* is a weakly Pareto optimal allocation, then there exists $\lambda^* = (\lambda_1^*, \dots, \lambda_m^*) \in \Delta$ such that x^* maximizes social welfare function on A.

(7) Let u_i is concave for every i. Using the utility set U, illustrate the case where x^* is a Pareto optimal allocation but there is no $\lambda^* \in R^m_{++}$ such that x^* maximizes social utility function on A.

3. Consider an economy $\mathcal{E} = \{(X_i, \succeq_i, e_i) : i = 1, \ldots, m\}$, where $e_i \in X_i = R^\ell_+$ for every i. Let $\succeq_i$ be complete, transitive, continuous, and monotone. Show that the set of Pareto optimal allocations is a compact subset of $R^{m\ell}$.

4. Consider the following economies. In homework 3, we found the competitive equilibria for each economy.

(a) Check the stability of each equilibrium.

(b) Find the set of Pareto optimal allocations.

(c) Find the set of individually rational allocations.

$$
\begin{array}{lllllll}
(1) & u_1(x,y) = x + y + xy & e_1 = (1,0) & (2) & u_1(x,y) = \min\{2x, y\} & e_1 = (2,8) \\
 & u_2(x,y) = xy & e_2 = (0,1) & & u_2(x,y) = \min\{x, 3y\} & e_2 = (6,0)
\end{array}
$$

$$
\begin{array}{lllllll}
(3) & u_1(x,y) = 2x + y, & e_1 = (0,1) & (4) & u_1(x,y) = \log x + y, & e_1 = (1,0) \\
 & u_2(x,y) = xy, & e_2 = (1,0). & & u_2(x,y) = \log x + 2y, & e_2 = (0,1).
\end{array}
$$

SOLUTION 4

1.

(1) ($\Rightarrow$) Let $x^* \in X$ be a Pareto optimal allocation. Suppose that x_i^* is not a solution to the problem for some i. Then there is $x_i \in X_i$ such that $u_i(x_i) > u_i(x_i^*)$ with $u_j(x_j) \geq u_j(x_j^*)$ for all $j \neq i$ and $\sum_i x_i \leq \sum_i e_i$. This is a contradiction to the Pareto optimality. ($\Leftarrow$) Let $x^* \in X$ be a solution to the problem. Suppose that x_i^* is not Pareto optimal. Then there is an allocation such that $u_i(x_i) > u_i(x_i^*)$ for some i and $u_j(x_j) \geq u_j(x_j^*)$ for all j where $\sum_i x_i \leq \sum_i e_i$. This contradicts that x^* is a solution to the problem. $\square$

(2) The Lagrangian is given by

$$L = u_i(x_i) + \sum_{j \neq i} \lambda_j [u_j(x_j) - u_j(x_j^*)] + \sum_{k=1}^{\ell} \mu_k \left[\sum_{i=1}^{m} (e_{ik} - x_{ik}) \right].$$

The first order condition is as follows.

$$\frac{\partial u_i(x_i^*)}{\partial x_{ik}} - \mu_k = 0, \; k = 1, \ldots, \ell, \tag{25}$$

$$\lambda_j \frac{\partial u_j(x_j^*)}{\partial x_{jk}} - \mu_k = 0, \; k = 1, \ldots, \ell, ; \forall j \neq i \tag{26}$$

If we consider a Pareto optimal allocation x^* as a Walrasian equilibrium allocation, we obtain the following through the utility optimization for every i,

$$\frac{\partial u_i(x_i^*)}{\partial x_{ik}} - \alpha_i p_k = 0, \; k = 1, \ldots, \ell.$$

Thus, if we take the shadow price μ_k of k-th good to be equal to Walrasian price p_k of k-th good for every k, then multiplier λ_j of the j-th utility constraint is equal to $1/\alpha_j$ which is the reciprocal of j-th consumer's marginal utility of income, *i.e.*, $\lambda_i = 1/\alpha_i$.

2.

(1) Let $x \in A$ and $x' \in A$. Since u_i is concave, $u_i(\alpha x_i + (1-\alpha)x_i') \geq \alpha u_i(x_i) + (1-$

$\alpha)u_i(x_i')$, for every $\alpha \in [0,1]$ and for every i. Thus, for every $\alpha \in [0,1]$,

$$\begin{aligned} W(\alpha x + (1-\alpha)x') &= \sum_{i=1}^{m} \lambda_i u_i(\alpha x_i + (1-\alpha)x_i') && (27) \\ &\geq \sum_{i=1}^{m} \lambda_i[\alpha u_i(x_i) + (1-\alpha)u(x_i')] && (28) \\ &= \alpha \sum_{i=1}^{m} \lambda_i u_i(x_i) + (1-\alpha)\sum_{i=1}^{m} u(x_i') && (29) \\ &= \alpha W(x) + (1-\alpha)W(x'). \ \square && (30) \end{aligned}$$

(2) Let $x \in A$ and $x' \in A$. For ever i, for every $\alpha \in [0,1]$, $u_i(\alpha x_i + (1-\alpha)x_i') \geq \alpha u_i(x_i) + (1-\alpha)u_i(x_i')$, and for some i, for every $\alpha \in (0,1)$ $u_i(\alpha x_i + (1-\alpha)x_i') > \alpha u_i(x_i) + (1-\alpha)u_i(x_i')$. Thus, for every $\alpha \in (0,1)$,

$$\begin{aligned} W(\alpha x + (1-\alpha)x') &= \sum_{i=1}^{m} \lambda_i u_i(\alpha x_i + (1-\alpha)x_i') && (31) \\ &> \sum_{i=1}^{m} \lambda_i[\alpha u_i(x_i) + (1-\alpha)u(x_i')] && (32) \\ &= \alpha W(x) + (1-\alpha)W(x'). \ \square && (33) \end{aligned}$$

(3) No. Consider one good economy with $u_i(x_i) = x_i^2$ for $i = 1,2$ and $\sum_i e_i = 3$. Let $x = (1,2)$, $x' = (2,1)$ and $\lambda_i = 1$ for $i = 1,2$. Then

$$W(x/2 + x'/2) = W(3/2, 3/2) = \frac{9}{2} < 5 = W(1,2) = \min\{W(x), W(x')\}.$$

(4) Let x^* maximizes $W(x) = \sum_i \lambda_i u_i(x_i)$ on A for some $\lambda \in R_+^m \setminus \{0\}$. Suppose that x^* is not weakly Pareto optimal. Then, there is a allocation $x' \in A$ such that $u_i(x_i') > u_i(x_i^*)$ for every i. Thus $\sum_i \lambda_i u_i(x_i) > \sum_i \lambda_i u_i(x_i^*)$, a contradiction. $\square$

(5) Let x^* maximizes $W(x) = \sum_i \lambda_i u_i(x_i)$ on A for some $\lambda \in R_{++}^m$. Suppose that x^* is not Pareto optimal. Then There is a allocation $x' \in A$ such that $u_i(x_i') \geq u_i(x_i^*)$ for every i and $u_i(x_i) > u_i(x_i^*)$ for some i. Thus $\sum_i \lambda_i u_i(x_i) > \sum_i \lambda_i u_i(x_i^*)$, a contradiction. $\square$

(6) Since u_i is concave for every i, the utility set U is convex. To verify this, choose $u \in U$ and $u' \in U$. Then there are $x \in A$ and $x' \in A$ such that $u_i = u_i(x_i)$ and $u_i' = u_i(x_i')$ for all i. Define $u(x) = (u_1(x_1), \ldots, u_m(x_m))$. For every $\alpha \in [0,1]$,

$$\begin{aligned} \alpha u + (1-\alpha)u' &= \alpha u(x) + (1-\alpha)u(x') && (34) \\ &\leq u(\alpha x + (1-\alpha)x'), && (35) \end{aligned}$$

since u_i is concave for every i. Because $\alpha x + (1-\alpha)x' \in A$, $\alpha u + (1-\alpha)u' \in U$. Let x^* be weakly Pareto optimal. Now define $P = \{u \in R^m_+ : u \gg u(x^*)\}$. Then P is clearly convex. Because $U \cap P = \emptyset$, by Separating Hyperplane Theorem, there is $\lambda^* \in R^\ell \setminus \{0\}$ such that $\lambda^* \cdot u' \geq \lambda^* \cdot u$ for every $u' \in P$ and $u \in U$. Note that $\lambda^* \geq 0$. Otherwise, the inequality cannot hold when we take a sufficiently large u_i corresponding to $\lambda^*_i < 0$. Because this inequality is homogeneous degree of zero with respect to λ^*, we can take $\lambda^* \in \Delta$. Consider $u^n = u(x^*) + (1/n, 1/n, \ldots, 1/n) \in P$ for every n. Then $\lambda^* \cdot u^n \geq \lambda^* \cdot u(x)$ for all $x \in A$. Since $u^n \to u(x^*)$, $\lambda^* \cdot u(x^*) \geq \lambda^* \cdot u(x)$ for all $x \in A$. That is, x^* maximizes $\sum \lambda^*_i u_i(x_i)$ on A. □

(7) Consider two-consumer one-good economy where $u_i(x_i) = \sqrt{x_i}$ and $e_i = i - 1$ for $i = 1, 2$. Then $U = \{u \in R^2_+ : (u_1)^2 + (u_2)^2 \leq 1\}$. Let $x^* = (0, 1)$. Then x^* is Pareto optimal. Now suppose that there is $\lambda^* \in R^m_{++}$ such that x^* maximizes $\sum_i \lambda^*_i u_i(x_i)$ on A. Without loss of generality, assume that $\lambda^*_1 \geq \lambda^*_2 > 0$. Then we have

$$W(1/2, 1/2) = \lambda^*_1 u_1(1/2) + \lambda^*_2 u_2(1/2) \geq \lambda^*_1 \sqrt{2} > \lambda^*_2 = \lambda^*_1 u_1(0) + \lambda^*_2 u_2(1) = W(x^*)$$

This is a contradiction. □

3. Let $P(\mathcal{E})$ be the set of Pareto optimal allocations of $\mathcal{E}$. Let $x \in P(\mathcal{E})$. Since x is feasible, $0 \leq x \leq (\sum_i e_i, \sum_i e_i, \ldots, \sum_i e_i)$. Thus, $P(\mathcal{E})$ is bounded. Now suppose that $P(\mathcal{E})$ is not closed. Then, there is a sequence $\{x^n\}$ in $P(\mathcal{E})$ such that $x^n \to x \notin P(\mathcal{E})$. Therefore, there is an allocation $x' \in A = \{x \in X : \sum_i x_i = \sum_i e_i\}$ such that $x'_i \succ_i x_i$ for some i and $x'_j \succeq_j x_j$ for all $j \neq i$. By continuity, for sufficiently small $\varepsilon > 0$, $x'_i - \epsilon \succ_i x_i$ with $\epsilon = (\varepsilon, \varepsilon, \ldots, \varepsilon)$. By monotonicity, $x'_j + \epsilon/(m-1) \succ_j x_j$ for all $j \neq i$. By continuity again, for sufficiently large n, $x'_i - \epsilon \succ_i x^n_i$ for some i and $x'_j + \epsilon/(m-1) \succ_j x^n_j$ for all $j \neq i$. The fact that

$$(x'_i - \epsilon) + \sum_{j \neq i} \left(x'_j + \frac{1}{m-1}\epsilon \right) = \sum_{i=1}^m x'_i = \sum_{i=1}^m e_i$$

contradicts that $x^n \in P(\mathcal{E})$. □

4. When we check the stability of the equilibria, we have only to examine whether the excess demand is increasing or decreasing around the equilibrium prices.

(1) (a) Around the equilibrium, the excess demands are given by

$$E_x = \frac{p_y - p_x}{p_x}, \tag{36}$$

$$E_y = \frac{p_x - p_y}{p_y}. \tag{37}$$

Hence the equilibrium is stable for p_x and is stable for p_y.

(b) The Lagrangian is given by

$$L = x_1 + y_1 + x_1 y_1 + \lambda_1(x_2 y_2 - \bar{u}) + \lambda_2(1 - x_1 - x_2) + \lambda_3(1 - y_1 - y_2).$$

Then we have the set of Pareto optimal allocation as follows.

$$\{((x_1, y_1), (x_2, y_2)) \in R^4_+ : y_1 = x_1,\ 0 \le x_1 \le 1,\ x_1 + x_2 = y_1 + y_2 = 1\}.$$

(c) The set of individually rational allocations is given by

$$\{((x_1, y_1), (x_2, y_2)) \in R^4_+ : x_1 + y_1 + x_1 y_1 \ge 1,\ x_2 y_2 \ge 0,\ x_1 + x_2 = y_1 + y_2 = 1\}.$$

(2) (a) Around the equilibrium, the excess demands are given by

$$E_x = \frac{2p_y(3p_x - 4p_y)}{3p_x^2 + 7p_x p_y + 2p_y^2}, \tag{38}$$

$$E_y = \frac{2p_x(4p_y - 3p_x)}{3p_x^2 + 7p_x p_y + 2p_y^2}. \tag{39}$$

Hence the equilibrium is unstable for p_x and is unstable for p_y.

(b) By the graph, we have the set of Pareto optimal allocation as follows.

$$\{((x_1, y_1), (x_2, y_2)) \in R^4_+ : x_1 + x_2 = y_1 + y_2 = 8,$$

$$\begin{array}{rcccll} 2x_1 & \le & y_1 & \le & (1/3)x_1 + (16/3) & \text{if } x_1 \in [0, 16/5), \\ (1/3)x_1 + (16/3) & \le & y_1 & \le & 2x_1 & \text{if } x_1 \in [16/5, 4), \\ (1/3)x_1 + (16/3) & \le & y_1 & \le & 8 & \text{if } x_1 \in [4, 8]\}. \end{array}$$

(c) The set of individually rational allocations is given by

$$\{((x_1, y_1), (x_2, y_2) \in R^4_+ : 2 \le x_1 \le 8,\ 4 \le y_1 \le 8,\ x_1 + x_2 = y_1 + y_2 = 8\}.$$

(3) (a) Around the equilibrium, the excess demands are given by

$$E_x = \begin{cases} \frac{2p_y - p_x}{2p_x} & \text{if } p_x \le 2p_y, \\ -\frac{1}{2} & \text{if } p_x > 2p_y, \end{cases} \tag{40}$$

$$E_y = \begin{cases} \frac{p_y - 2p_x}{2p_x} & \text{if } p_x \le 2p_y, \\ \frac{p_y}{2p_x} & \text{if } p_x > 2p_y. \end{cases} \tag{41}$$

$$\tag{42}$$

Hence the equilibrium is stable for p_x and is unstable for p_y.

(b) The Lagrangian is given by

$$L = 2x_1 + y_1 + \lambda_1(x_2y_2 - \bar{u}) + \lambda_2(1 - x_1 - x_2) + \lambda_3(1 - y_1 - y_2).$$

Then we have the set of Pareto optimal allocation as follows.

$$\begin{aligned} \{((x_1, y_1), (x_2, y_2)) \in R^4_+ : y_1 &= 0 \quad \text{if } x \in [0, 1/2); \\ y_1 &= 2x_1 - 1 \quad \text{if } x_1 \in [0, 1],\ x_1 + x_2 = y_1 + y_2 = 1\}. \end{aligned}$$

(c) The set of individually rational allocations is given by

$$\{((x_1, y_1), (x_2, y_2) \in R^4_+ : 2x_1{+}y_1 \geq 1,\ x_1 \leq 1,\ 0 \leq y_1 \leq 1,\ x_1{+}x_2 = y_1{+}y_2 = 1\}$$

(4) (a) Around the equilibrium, the excess demands are given by

$$E_x = \frac{3p_y - 2p_x}{2p_x} \tag{43}$$

$$E_y = \frac{2p_x - 3p_y}{2p_y}. \tag{44}$$

Hence the equilibrium is stable for p_x and is stable for p_y.

(b) The Lagrangian is given by

$$L = \log x_1 + y_1 + \lambda_1(\log x_2 + 2y_2 - \bar{u}) + \lambda_2(1 - x_1 - x_2) + \lambda_3(1 - y_1 - y_2).$$

Then we have the set of Pareto optimal allocation as follows.

$$\begin{aligned} \{((x_1, y_1), (x_2, y_2)) \in R^4_+ : y_1 &= 0 \quad \text{if } x \in [0, 2/3); \\ y_1 &\in [0, 1) \quad \text{if } x_1 = 2/3; \\ y_1 &= 1 \quad \text{if } x_1 \in [2/3, 1],\ x_1 + x_2 = y_1 + y_2 = 1\}. \end{aligned}$$

(c) The set of individually rational alloctions is given by

$$\{((x_1, y_1), (x_2, y_2) \in R^4_+ : \log x_1{+}y_1 \geq 0,\ \log x_2{+}2y_2 \geq 0,\ x_1{+}x_2 = y_1{+}y_2 = 1\}.$$

HOMEWORK 5

♠ We consider an exchange economy $\mathcal{E} = \{(X_i, \succeq_i, e_i) : i = 1, 2, \ldots, m\}$ where $0 \neq e_i \in X_i = R_+^\ell$ for every i and $\succeq_i$ is complete and transitive, when appropriate.

1. Consider a one agent exchange economy $\mathcal{E} = \{(X, \succeq, e)\}$ where $e \in X = R_+^\ell$.

(1) State the minimal conditions to ensure that any Pareto optimal allocation x^* is supported by a price $p^* \in R^\ell \setminus \{0\}$.

(2) Prove the statement.

(3) Show that (p^*, x^*) is a quasi-equilibrium.

(4) Is it a Walrasian equilibrium ?

2. Prove that if $\succeq_i$ is locally nonsatiated for every i, then a Walrasian equilibrium allocation is Pareto optimal. What if the local nonsatiation assumption is violated ?

3. Show that if $\succeq_i$ is strongly monotone for every i, then a Walrasian equilibrium is a quasi-equilibrium. Can the strong monotonicity be relaxed to the monotonicity ?

4. Show that a Walrasian equilibrium allocation which is a quasi-equilibrium allocation is Pareto optimal.

5. By the first welfare theorem, we know that if $\succeq_i$ is strictly convex for every i, then a Walrasian equilibrium allocation is Pareto optimal. Under the same assumptions, can we say that a Walrasian equilibrium is a quasi-equilibrium ?

6. Consider an agent i and suppose that $\succeq_i$ is continuous and monotone. Suppose that $x_i \in X_i$ and $p \in R_+^\ell \setminus \{0\}$ are such that $p \cdot x_i > 0$, and $x_i' \succ_i x_i$ implies $p \cdot x_i' \geq p \cdot x_i$. Prove that $x_i' \succ_i x_i$ implies $p \cdot x_i' > p \cdot x_i$. What if X_i is not convex ? What if $\succeq_i$ is not continuous ? What if $p \cdot x_i = 0$?

7. Let $\succeq_i$ be continuous for every i. Show that a quasi-equilibrium with $p \gg 0$ is a Walrasian equilibrium.

8. Suppose that $\succeq_i$ is continuous and strongly monotone for every i. Show that if (p, x) is a quasi-equilibrium with $x_i \in int R_+^\ell$ for some i, then $p \gg 0$.

SOLUTION 5

1.

(1) $\succeq$ is nonsatiated and convex ($x' \succ x \Rightarrow \alpha x' + (1-\alpha)x \succ x$ for $\alpha \in (0,1)$ and $x' \sim x \Rightarrow \alpha x' + (1-\alpha)x \succeq x$).

(2) Let an allocation x^* be Pareto optimal. Let $P(x^*) = \{x \in X : x \succ x^*\}$. Since x^* is Pareto optimal, there is no $x \in X$ such that $x = e$ and $x \succ x^*$. Thus, $e \notin P(x^*)$. Moreover, $P(x^*)$ is nonempty by nonsatiation and is convex since $\succeq$ is complete, transitive, and convex. By Separating Hyperplane Theorem, there is $p^* \in R^\ell \setminus \{0\}$ such that for every $x \in P(x^*)$, $p^* \cdot x \geq p \cdot e$. Because x^* is feasible, *i.e.*. $x^* = e$, we conclude that for every $x \succ x^*$, $p^* \cdot x \geq p^* \cdot x^*$. Let $x \sim x^*$. Then by nonsatiation, we have $\hat{x} \in X$ such that $\hat{x} \succ x$. By convexity, $x^\alpha = \alpha\hat{x} + (1-\alpha)x \succ x$ for $\alpha \in (0,1]$. By transitivity, $x^\alpha \succ x^*$ so that $p^* \cdot x^\alpha \geq p \cdot x^*$. By letting $\alpha \to 0$, we have $p^* \cdot x \geq p^* \cdot x^*$. □

(3) Because $x^* = e$, it is a quasi-equilibrium.

(4) Not necessarily. Consider $X = R^2_+$ and $\succeq$ is such that $x' \succ x$ iff $x'_1 > x_1$ and $x' \sim x$ iff $x'_1 = x_1$. Let $x^* = e = (1,0)$ and $p^* = (0,1)$. Then the statement is satisfied but this is not maximizing his preference.

2.

(1) Let (p^*, x^*) be a Walrasian equilibrium. Suppose x^* is not Pareto optimal. Then there is a feasible allocation $x \in X$ such that $x_j \succ_j x_j^*$ for some j and $x_i \succeq_i x_i^*$ for all $i \neq j$. Therefore, $p^* \cdot x_j > p^* \cdot e_j$ for some j. However, the local nonsatiation implies that there is $x_i^n \in B_{1/n}(x_i)$ such that $x_i^n \succ_i x_i$ for every $i \neq j$. Therefore, by transitivity, $x_i^n \succ_i x_i^*$ so that $p^* \cdot x_i^n > p^* \cdot x^*$. As $n \to \infty$, we have $p^* \cdot x_i \geq p^* \cdot e_i$ for every $i \neq j$. Consequently, $\sum_{i=1}^m p^* \cdot x_i > \sum_{i=1}^m p^* \cdot e_i$, which violates the feasibility and leads to a contradiction. □

(2) When the local nonsatiation is not satisfied, we have an example where a Walrasian equilibrium allocation may not be Pareto optimal. It can be easily illustrated by a two-agent economy where one consumer has a thick indifference set.

3.

(1) Let (p^*, x^*) is a Walrasian equilibrium. Consider $x_i \succeq_i x_i^*$ for every i. Then by strong monotonicity, $x_i + \varepsilon^n \succ x_i^*$ with $\varepsilon^n = (1/n, 0, \ldots, 0)$ for every n and

for every i. Since x^* is a Walrasian equilibrium allocation, $p^* \cdot x_i > p^* \cdot e_i$. As $n \to \infty$, $p^* \cdot x_i \geq p^* \cdot e_i$. $\square$

(2) Yes. The proof is the same as above except $\varepsilon^n = (1/n, 1/n, \ldots, 1/n)$. $\square$

4. Let (p^*, x^*) be a Walrasian equilibrium. Suppose x^* is not Pareto optimal. Then there is a feasible allocation $x \in X$ such that $x_j \succ x_j^*$ for some j and $x_i \succeq_i x_i^*$ for all $i \neq j$. Since x^* is a Walrasian equilibrium allocation, $p^* \cdot x_j > p^* \cdot e_j$ for some j. However for every $i \neq j$, $p^* \cdot x_i \geq p^* \cdot x^*$ since x^* is a quasi-equilibrium allocation. Consequently, $\sum_{i=1}^m p^* \cdot x_i > \sum_{i=1}^m p^* \cdot e_i$, which violates the feasibility and leads to a contradiction. $\square$

5. Yes. Let (p^*, x^*) be a Walrasian equilibrium. Now suppose that $x_i \succeq_i x_i^*$ for every i. If $x_i \succ_i x_i^*$, we have $p^* \cdot x_i > p^* \cdot x_i^*$. If $x_i \sim_i x_i^*$, then by strict convexity, we have $x_i^\alpha = \alpha x_i + (1-\alpha)x^* \succ_i x_i^*$ for $\alpha \in (0,1)$. Since (p^*, x^*) is a Walrasian equilibrium, $p^* \cdot x_i^\alpha > p^* \cdot x_i^*$. As $\alpha \to 1$, $p^* \cdot x_i \geq p^* \cdot x_i^*$. Consequently, we conclude that (p^*, x^*) is a quasi-equilibrium. $\square$

6.

(1) Suppose that $p \cdot x_i' = p \cdot x_i$. Since $p \in R_+^\ell \setminus \{0\}$ and $p \cdot x_i > 0$, there is $\hat{x}_i \in X_i$ such that $p \cdot \hat{x}_i < p \cdot x_i$. In particular, there is a k such that $p_k > 0$ and $x_{ik} > 0$. Then take $\hat{x}_i$ as x_i except replacing k-th consumption with $x_{ik} - \varepsilon > 0$. However, by the continuity of $\succeq_i$ and the convexity of X_i, we have $x_i^\alpha = (1-\alpha)x_i' + \alpha\hat{x}_i \succ_i x_i$ for sufficiently small $\alpha > 0$ and $x_i^\alpha \in X_i$. So that $p \cdot x_i^\alpha \geq p \cdot x_i$. But by our assumption, we have $p \cdot x_i^\alpha = \alpha p \cdot x_i' + (1-\alpha)p \cdot \hat{x}_i < p \cdot x_i$ for $\alpha > 0$. This is a contradiction. $\square$

(2) Let $X_i = [R_+ \times \{0\}] \cup [R_+ \times \{1\}]$ and $\succeq_i$ be such that $x' \succ_i x$ if $x_1' > x_1$ with $x_2' = x_2$, and $(0,1) \succ_i (x_1, 0)$ for every $x_1 \in R_+$. Consider $x_i = (1,0)$, $x_i' = (0,1)$, and $p = (1,1)$, Then $x_i' \succ_i x_i$ but $p \cdot x_i' = p \cdot x_i$.

(3) Consider a lexicographic preference in R_+^2. Let $x_i = (1,1)$, $x_i' = (1,2)$, and $p = (1,0)$. Then $x_i' \succ_i x_i$ but $p \cdot x_i' = p \cdot x_i$.

(4) Consider the case of Problem 1 (4). Let $x_i' = (2,0) \succ_i (1,0) = x_i$. Then $p \cdot x_i' = p \cdot x_i$.

7. Let (p, x) be a quasi-equilibrium. Since $\succeq_i$ is reflexive for every i, it follows that $p \cdot x_i = p \cdot e_i$ for every i. For every i, let $x_i' \succ_i x_i$. Then $p \cdot x_i' \geq p \cdot e_i = p \cdot x_i$ for every i. Because $p \cdot x_i > 0$, Problem 6 implies that $p \cdot x_i' > p \cdot x_i = p \cdot e_i$ for every i. Hence (p, x) is a Walrasian equilibrium. $\square$

8. For some i, we have $x_i \in intX_i$ so that $p \cdot x_i > 0$. Since $\succeq_i$ is reflexive, it follows that $p \cdot x_i = p \cdot e_i$. Let $x_i' = x_i + \varepsilon^k$ where $\varepsilon^k = (0, \ldots, 0, 1, 0, \ldots, 0)$ with k-th element 1. By strong monotonicity, $x_i' \succ_i x_i$. Since (p, x) is a quasi-equilibrium, $p \cdot (x_i + \varepsilon^k) \geq p \cdot e_i = p \cdot x_i > 0$. By Problem 6 implies that $p_k = p \cdot \varepsilon^k > 0$. This holds for every $k = 1, \ldots, \ell$, so that $p \gg 0$. $\square$

HOMEWORK 6

♠ Consider an economy $\mathcal{E} = \{(X_i, \succeq_i, e_i) : i = 1, \ldots, m\}$ where $e_i \in X_i = R^\ell_+$ for every i and $\succeq_i$ is complete and transitive for every i. Define

- $W(\mathcal{E})$: the set of Walrasian equilibria of $\mathcal{E}$.
- $W^*(\mathcal{E})$: the set of Walrasian equilibrium allocations of $\mathcal{E}$.
- $P(\mathcal{E})$: the set of Pareto optimal allocations of $\mathcal{E}$.
- $WP(\mathcal{E})$: the set of weakly Pareto optimal allocations of $\mathcal{E}$.
- $IR(\mathcal{E})$: the set of individually rational allocations of $\mathcal{E}$.
- $C(\mathcal{E})$: the set of core allocations of $\mathcal{E}$.
- $V(\mathcal{E})$: the set of value allocations of $\mathcal{E}$.

1. Show that if $\succeq_i$ is continuous for every i, $C(\mathcal{E})$ is compact.

2. Show that if $m = 2$, $C(\mathcal{E}) = WP(\mathcal{E}) \bigcap IR(\mathcal{E})$.

3. Show that $W^*(\mathcal{E}) \subset C(\mathcal{E})$.

4. Let us replicate the economy $\mathcal{E}$ by n-times and call it as n-fold replica economy $\mathcal{E}^n$. Then, there are $n \cdot m$ consumers and n identical consumers. Denote the i-th type consumer generated by j-th replication by "ij". Let $\succeq_i$ be continuous, strictly monotonic, and strictly convex for every i.

(1) Show that if $x^* \in C(\mathcal{E}^n)$, then $x^*_{ij} = x^*_{ij'}$ for every i, j and j'.

(2) Define $C^n(\mathcal{E}) = \{x \in \prod_{i=1}^m X_i : x_i = x^*_{ij},\ \forall i,\ \text{where } x^* \in C(\mathcal{E}^n)\}$. Show that $C^{n+1}(\mathcal{E}) \subset C^n(\mathcal{E})$ for every n.

(3) Let $E(\mathcal{E}) = \bigcap_{n=1}^\infty C^n(\mathcal{E})$. Assume that $C(\mathcal{E}^n) \neq \emptyset$ for every n. Show that $E(\mathcal{E}) \neq \emptyset$.

(4) Show that $W^*(\mathcal{E}) \subset E(\mathcal{E})$.

(5) (Optional) Show that $E(\mathcal{E}) \subset W^*(\mathcal{E})$ if $e_i \in intX_i$ for every $i \in I$

♠ Consider an economy $\mathcal{E} = \{(X_i, u_i, e_i) : i = 1, \ldots, m\}$ where $e_i \in X_i = R^\ell_+$ for every i and $u_i : X_i \to R$ is a utility function for every i.

5. Show that if x^* is a value allocation with respect to $\lambda^* \in R^m_+ \setminus \{0\}$, then it is a value allocation with respect to $\alpha\lambda^*$ for all $\alpha > 0$.

6. Show that $V(\mathcal{E}) \subset WP(\mathcal{E})$.

7. Show that $V(\mathcal{E}) \subset P(\mathcal{E})$ where $\lambda \in R^m_{++}$.

8. Show that $V(\mathcal{E}) \subset IR(\mathcal{E})$ where $\lambda \in R^m_{++}$. **N. B.** It follows from Problems 6 and 8 that if $m = 2$, $V(\mathcal{E}) \subset C(\mathcal{E})$ where $\lambda \in R^2_{++}$.

9. Consider two economies, each of which is composed of two goods and two consumers as follows.

$$
\begin{array}{llllll}
(a) & u_1(x,y) = x + y, & e_1 = (2,1), & (b) & u_1(x,y) = \min\{2x, y\}, & e_1 = (2,1), \\
 & u_2(x,y) = \min\{x, y\}, & e_2 = (1,2). & & u_2(x,y) = \min\{x, 2y\}, & e_2 = (1,2).
\end{array}
$$

(1) Find $P(\mathcal{E})$ for each economy.

(2) Find $C(\mathcal{E})$ for each economy.

(3) Find $W(\mathcal{E})$ for each economy.

(4) Find the value allocation for each economy where $\lambda_1 = 1/2, \lambda_2 = 1$.

10. Consider an economy $\mathcal{E} = \{(X_i, u_i, e_i) : i = 1, 2, 3\}$ where $X_i = R^2_+$ and

$$
\begin{aligned}
u_1(x,y) &= \min\{x, 2y\}, & e_1 &= (4,0) \\
u_2(x,y) &= \min\{2x, 4y\}, & e_2 &= (0,2) \\
u_3(x,y) &= (x + 2y)/2, & e_3 &= (0,0)
\end{aligned}
$$

(1) Find $C(\mathcal{E})$.

(2) Find $W(\mathcal{E})$.

(3) Find the value allocation for $\lambda_1 = \lambda_3 = 1$ and $\lambda_2 = 1/2$.

(5) For the value allocation in (4), is there a coalition who can lie about their preferences or initial endowments and become better off ? Give an example and calculate the resulting manipulated value allocation.

SOLUTION 6

1. Since any core allocation is feasible, $C(\mathcal{E})$ is bounded in the same way as $W(\mathcal{E})$. Suppose that $C(\mathcal{E})$ is not closed. Then there is a sequence x_n of core allocations which converges to $x^* \notin C(\mathcal{E})$. Then there is a coalition S and $(x_i)_{i\in S} \in \prod_{i\in S} X_i$ such that $\sum_{i\in S} x_i = \sum_{i\in S} e_i$ and $x_i \succ_i x_i^*$ for every $i \in S$. By continuity of $\succeq_i$, for every $i \in S$, we have $x_i \succ_i x_i^n$ for sufficiently large n. This contradicts that $x^n \in C(\mathcal{E})$. $\Box$

2. In the definition of core, take $S = \{1,2\}$ to get $C(\mathcal{E}) \subset WP(\mathcal{E})$. On the other hand, take $S = \{1\}$ or $\{2\}$ and get $C(\mathcal{E}) \subset IR(\mathcal{E})$. This proves that $C(\mathcal{E}) \subset WP(\mathcal{E}) \bigcap IR(\mathcal{E})$. To prove the reverse inclusion, suppose $x \in WP(\mathcal{E}) \bigcap IR(\mathcal{E})$ but $x \notin C(\mathcal{E})$. Then there is a coalition $S \subset \{1,2\}$ and $(x_i')_{i\in S} \in \prod_{i\in S} X_i$ such that $\sum_{i\in S} x_i' = \sum_{i\in S} e_i$ and $x_i' \succ_i x_i$ for every $i \in S$. If $S = \{i\}$, then $e_i \succ_i x_i$, *i.e.*, x is not individually rational, a contradiction. If $S = \{1,2\}$, x is not weakly Pareto optimal, a contradiction. $\Box$

3. Suppose not. Then there is $(p^*, x^*) \in W(\mathcal{E})$ but $x^* \notin C(\mathcal{E})$. Thus there is a coalition S and $(x_i)_{i\in S} \in \prod_{i\in S} X_i$ such that $\sum_{i\in S} x_i = \sum_{i\in S} e_i$ and $x_i \succ_i x_i^*$ for every $i \in S$. Since (p^*, x^*) is a Walrasian equilibrium, $p^* \cdot x_i > p^* \cdot e_i$ for every $i \in S$. Therefore, $p^* \cdot \sum_{i\in S} x_i > p^* \cdot \sum_{i\in S} e_i$, a contradiction. $\Box$

4.

(1) Suppose not. Then there is $x^* \in C(\mathcal{E}^n)$ such that $x_{kj}^* \neq x_{kj'}^*$ for some k, j, and j'. Now let the index $j_i \in \{1, \ldots, n\}$ is chosen for every i such that $x_{ij}^* \succeq_i x_{ij_i}^*$ for all j. Consider a coalition $S = \{1j_1, 2j_2, \ldots, mj_m\}$. Let $\bar{x}_i = (1/n)\sum_{j=1}^n x_{ij}^*$ for every i. Then $\bar{x}_i \succeq_i x_{ij_i}^*$ for every i and by strict convexity of $\succeq_k$, $\bar{x}_k \succ_k x_{kj_k}^*$. By continuity of $\succeq_k$, there is $\varepsilon \in R_+^\ell \setminus \{0\}$ such that $\bar{x}_k' = \bar{x}_k - \varepsilon \succ_k x_{kj_k}^*$. Then by strictly monotonicty of $\succeq_i$, $\bar{x}_i' = \bar{x}_i + [\varepsilon/(n-1)] \succ_i x_{ij_i}^*$ for every $i \neq k$. Finally, $(\bar{x}_i')_{i\in S}$ is feasible in S :

$$\sum_{i=1}^m \bar{x}_i' = \sum_{i=1}^m \left[\frac{1}{n}\sum_{j=1}^n x_{ij}^*\right] = \frac{1}{n}\sum_{i=1}^m\sum_{j=1}^n x_{ij}^* = \frac{1}{n}\sum_{i=1}^m (n \cdot e_i) = \sum_{i=1}^m e_i$$

Hence the coalition S and $(\bar{x}_i')_{i\in S}$ blocks the allocation x^*, a contradiction. $\Box$

(2) It is immediate since a coalition of n-replica economy is a coalition of $(n+1)$-replica economy. $\Box$

(3) We know that $C^n(\mathcal{E})$ is nonemtpy and compact for every n. Moreover, $\{C^n(\mathcal{E})\}$ has a finite intersection property since it is a decreasing sequence. Hence $E(\mathcal{E}) = \bigcap_{n=1}^\infty C^n(\mathcal{E})$ is nonempty. $\Box$

(4) Let $x \in W^*(\mathcal{E})$. Now choose any replica economy $\mathcal{E}^n$. Then define $x^*_{ij} = x_i$ for every i and j so that $x^* \in W^*(\mathcal{E}^n) \subset C(\mathcal{E}^n)$. By definition, $x \in C^n(\mathcal{E})$. This hold for every n. Hence $W^*(\mathcal{E}) \subset E(\mathcal{E})$. □

(5) Let $x^* \in E(\mathcal{E})$. Define $P_i(x^*_i) = \{z_i \in R^\ell : z_i + e_i \succ_i x^*_i\}$. Then $P(x^*) = co\,(\bigcup_{i=1}^m P_i(x^*_i))$ is nonempty, open, and convex. We claim that $0 \notin P(x^*)$. Suppose by way of contradiction that $0 \in P(x^*)$. Then there is $z_i \in P_i(x^*_i)$ for every i and $\alpha \in \Delta$ such that $\sum_{i=1}^m \alpha_i z_i = 0$. For every n, let λ^n_i be the smallest integer greater than or equal to $n\alpha_i$. For every i such that $\alpha_i > 0$, define $z^n_i = (n\alpha_i/\lambda^n_i)z_i$. Now $z^n_i + e_i \in X_i$ since $z^n_i + e_i = t_n(z_i + e_i) + (1 - t_n)e_i$ where $t_n = (n\alpha_i/\lambda^n_i)$. Since $\succeq_i$ is continuous for every i and $t_n \to 1$, for sufficiently large n, we have $z^n_i + e_i \succ_i x^*_i$. Then a coalition of λ^n_i traders of type i blocks the allocation x^* since

$$\sum_{i=1}^m \lambda^n_i z^n_i = n \sum_{i=1}^m \alpha_i z_i = 0.$$

This contraditiction implies that $0 \notin P(x^*)$. By the Separating Hyperplane Theorem, there is a $p^* \in R^\ell \setminus \{0\}$ such that $p^* \cdot z \geq 0$ for every $z \in P(x^*)$. Now let $x'_i \succ_i x^*_i$. Then $x'_i - e_i \in P_i(x^*_i)$ so that $p^* \cdot x'_i \geq p^* \cdot e_i$. Since $\succeq_i$ is continuous and $e_i \in int X_i$, $p^* \cdot x'_i > p^* \cdot e_i$. Furthermore, by strict monotonicity, we have $p^* \cdot x^*_i = p^* \cdot e_i$. This shows that $x^* \in W(\mathcal{E})$. □

5. For every $S \subset I$, $V_{\lambda^*}(S) = \max\{\sum_{i\in S} \lambda^*_i u_i(x_i) : \sum_{i\in S} x_i = \sum_{i\in S} e_i\}$. Therefore, $V_{\alpha\lambda^*}(S) = \alpha V_{\lambda^*}(S)$ for every S. Now note that the Shapley value $\Psi_i(V_{\lambda^*})$ is linear in V_{λ^*}. Thus we have $\Psi_i(V_{\alpha\lambda^*}) = \Psi_i(\alpha V_{\alpha\lambda^*}) = \alpha\Psi_i(V_{\lambda^*})$. This implies that $\Psi_i(V_{\alpha\lambda^*}) = (\alpha\lambda^*_i)u_i(x_i)$ iff $\Psi_i(V_{\lambda^*}) = \lambda^*_i u_i(x_i)$. Hence, we have the same value allocation. □

6. Suppose that $x^* \in V(\mathcal{E})$ but $x^* \notin WP(\mathcal{E})$. There is an allocation $x \in X$ such that $u_i(x_i) > u_i(x^*_i)$ for every i and $\sum_{i\in I} x_i = \sum_{i\in I} e_i$. Therefore, $\sum_{i\in I} \lambda^*_i u_i(x_i) > \sum_{i\in I} \lambda^*_i u_i(x^*_i) = \sum_{i\in I} \Psi_i(V_\lambda) = V_\lambda(I)$, a contradiction to the definition of $V_\lambda(I)$. □

N.B. Note that

$$\sum_{i\in I} \Psi_i(V) = \sum_{i\in I} \sum_{S\subset I, \ni i} \frac{(|S|-1)!(|I|-|S|)!}{|I|!}[V_\lambda(S) - V_\lambda(S \setminus \{i\})].$$

For every proper coalition $S(\neq I)$, its coefficient in the sum is given by

$$|S|\frac{(|S|-1)!(|I|-|S|)!}{|I|!} - \left[(|I|-|S|)\frac{|S|!(|I|-|S|-1)!}{|I|!}\right] = 0.$$

Hence $\sum_{i\in I} \Psi_i(V_\lambda) = V_\lambda(I)$.

7. Suppose that $x^* \in V(\mathcal{E})$ but $x^* \notin P(\mathcal{E})$. There is an allocation $x \in X$ such that $u_i(x_i) \geq u_i(x_i^*)$ for every i, $u_i(x_i) > u_i(x_i^*)$ for some i, and $\sum_{i\in I} x_i = \sum_{i\in I} e_i$. Therefore, $\sum_{i=1}^m \lambda_i^* u_i(x_i) > \sum_{i=1}^m \lambda_i^* u_i(x_i^*) = \sum_{i\in I} \Psi_i(V_\lambda) = V_\lambda(I)$, a contradiction to the definition of $V_\lambda(I)$. □

8. Let $x \in V(\mathcal{E})$. Suppose $x \notin IR(\mathcal{E})$. Then there is an agent i such that $u_i(e_i) > u_i(x_i)$. Then $\lambda_i u_i(e_i) > \lambda_i u_i(x_i) = \Psi_i(V_\lambda) \geq V_\lambda(\{i\}) = \lambda_i u_i(e_i)$, a contradiction. □

N.B. By superadditivity of V_λ, we have $V_\lambda(S) - V_\lambda(S \setminus \{i\}) \geq V_\lambda(\{i\})$. Hence

$$\Psi_i(V_\lambda) \geq \sum_{S\subset I, \ni i} \frac{(|S|-1)!(|I|-|S|)!}{|I|!} V_\lambda(\{i\}) = V_\lambda(\{i\}).$$

9.

(1) (a) $P(\mathcal{E}) = \{((x_1, y_1), (x_2, y_2)) \in R_+^4 : y_1 = x_1,\ x_1 + x_2 = y_1 + y_2 = 3\}$. (b) $P(\mathcal{E}) = \{((x_1, y_1), (x_2, y_2)) \in R_+^4 : 2x_1 \leq y_1 \leq (1/2)x_1 + 3/2$ with $x_1 \in [0, 1]; (1/2)x_1 + 3/2 \leq y_1 \leq 2x_1$ with $x_1 \in [1, 3/2]; (1/2)x_1 + 3/2 \leq y_1 \leq 3$ with $x_1 \in [3/2, 3],\ x_1 + x_2 = y_1 + y_2 = 3\}$.

(2) (a) $C(\mathcal{E}) = \{((x_1, y_1), (x_2, y_2)) \in R_+^4 : y_1 = x_1$ with$\}$ $x_1 \in [3/2, 2],\ x_1 + x_2 = y_1 + y_2 = 3\}$. (b) $C(\mathcal{E}) = \{((x_1, y_1), (x_2, y_2)) \in R_+^4 : 2x_1 \leq y_1 \leq (1/2)x_1 + 3/2$ with $x_1 \in [1/2, 1]; (1/2)x_1 + 3/2 \leq y_1 \leq 2x_1$ with $x_1 \in [1, 5/4]; (1/2)x_1 + 3/2 \leq y_1 \leq 5/2$ with $x_1 \in [5/4, 2],\ x_1 + x_2 = y_1 + y_2 = 3\}$.

(3) (a) $W(\mathcal{E}) = \{((p_x, p_y), (x_1, y_1), (x_2, y_2)) = ((1/2, 1/2), (3/2, 3/2), (3/2, 3/2)\}$. (b) $W(\mathcal{E}) = \{((p_x, p_y), (x_1, y_1), (x_2, y_2)) = ((1/2, 1/2), (1, 2), (2, 1))\}$.

(4) (a) $V(\{1\}) = 3/2,\ V(\{2\}) = 1,\ V(\{1, 2\}) = 3$. Then $\Psi_1(V) = 7/4,\ \Psi_2(V) = 5/4$. By th definition of value allocation, it must hold $(1/2)(x_1 + x_2) = 7/4$ and $\min\{x_2, y_2\} = 5/4$. Hence, $((7/4, 7/4), (5/4, 5/4))$ is the unique value allocation. (b) $V(\{1\}) = 1/2,\ V(\{2\}) = 1,\ V(\{1, 2\}) = 3$. Then $\Psi_1(V) = 5/4, \Psi_2(V) = 7/4$. It must hold that $(1/2)\min\{2x_1, y_1\} = 5/4$ and $\min\{x_2, 2y_2\} = 7/4$. Hence, there is no value allocation.

10.

(1) $C(\mathcal{E}) = \{((x_1, y_1), (x_2, y_2), (0, 0)) \in R_+^6 : y_1 = (1/2)x_1 + 1, x_1 + x_2 = 4,\ y_1 + y_2 = 2\}$.

(2) Equilibrium allocations will be on the diagonal. Any p is an equilibrium one.

(3) $V(\{i\}) = 0$ for every i and

$$V(\{1, 2\}) = V(\{1, 2, 3\}) = 4,\ V(\{1, 3\}) = V(\{2, 3\}) = 2.$$

Therefore, we have

$$\Psi_1(V) = \Psi_2(V) = 5/3, \ \Psi_3(V) = 2/3.$$

By the defintion of value allocation, it must hold that

$$\begin{aligned} \min\{x_1, 2y_1\} &= 5/3 \\ (1/2)\min\{2x_2, 4y_2\} &= 5/3 \\ (x_3 + 2y_3)/2 &= 2/3, \end{aligned}$$

which implies that $((5/3, 5/6), (5/3, 5/6), (2/3, 1/3))$ is the unique value allocation.

(4) (Preferences) Agent 1 and 2 can form a coalition and lie their preferences to give nothing to agent 3 : $u_1'(x, y) = (x + 2y)/2, \ u_2'(x, y) = x + 2y$. Then

$$\begin{aligned} &V'(\{1\}) = V'(\{2\}) = V'(\{1,3\}) = V'(\{2,3\}) = 2, \\ &V'(\{3\}) = 0, \ V'(\{1,2\}) = V'(\{1,2,3\}) = 4, \end{aligned}$$

Therefore, we have

$$\Psi_1'(V') = \Psi_2'(V') = 2, \ \Psi_3'(V') = 0.$$

By the defintion of value allocation, it must hold that

$$\begin{aligned} (x_1 + 2y_1)/2 = 2 \\ (x_2 + 2y_2)/2 = 2 \\ (x_3 + 2y_3)/2 = 0 \end{aligned}$$

which implies that $((2, 1), (2, 1), (0, 0))$ is a value allocation. Note that $u_1((2, 1)) > u_1((5/3, 5/6)$ and $u_2((2, 1)) > u_2((5/3, 5/6))$. (Endowments) Agent 1 and 2 can form a coalition and lie their preference to give nothing to agent 3 : $e_1' = (2, 1)$ $e_2' = (2, 1)$. Then

$$\begin{aligned} &V'(\{1\}) = V'(\{2\}) = V'(\{1,3\}) = V'(\{2,3\}) = 2, \\ &V'(\{3\}) = 0, \ V'(\{1,2\}) = V'(\{1,2,3\}) = 4, \end{aligned}$$

Therefore, we have

$$\Psi_1'(V) = \Psi_2(V) = 2, \ \Psi_3(V) = 0.$$

By the defintion of value allocation, it must hold that

$$\begin{aligned} \min\{x_1, 2y_1\} = 2 \\ (1/2)\min\{2x_2, 4y_2\} = 2 \\ (x_3 + 2y_3)/2 = 0 \end{aligned}$$

which implies that $((2, 1), (2, 1), (0, 0))$ is the unique value allocation. Note that $u_1((2, 1)) > u_1((5/3, 5/6))$ and $u_2((2, 1)) > u_2((5/3, 5/6))$.

HOMEWORK 7

♠ We consider the same economy as in Homework 6 and keep the notations. In addition, we define

- $F(\mathcal{E})$: the set of fair allocations.
- $CF(\mathcal{E})$: the set of coalitionally fair allocations.

1. Let $(p^*, x^*) \in W(\mathcal{E})$. Show that if $p^* \cdot x_i^* = p^* \cdot x_j^*$ for ever i and j, then x^* is envy-free.

2. Show that $W(\mathcal{E}) \subset CF(\mathcal{E})$.

3. Show that $CF(\mathcal{E}) \subset C(\mathcal{E})$.

4. Consider an economy $\mathcal{E} = \{(X_i, u_i, e_i) : i = 1, 2, 3\}$ where $X_i = R^2_+$ for every$i$ and

$$\begin{aligned} u_1(x, y) &= \min\{x, y\}, \quad e_1 = (1, 0) \\ u_2(x, y) &= \min\{x, y\}, \quad e_2 = (0, 1) \\ u_3(x, y) &= (x + y)/2, \quad e_3 = (0, 0) \\ u_4(x, y) &= (x + y)/2, \quad e_4 = (0, 0) \end{aligned}$$

(1) Find $C(\mathcal{E})$.

(2) Find $W(\mathcal{E})$.

(3) Find the value allocation where $\lambda_i = 1$ for $i = 1, 2, 3$ and $\lambda_4 = 0$.

(4) Show that the value allocation in (3) does not have the equal treatment property.

(5) Show that the value allocation in (3) is not coalitionally fair.

5. Show that for a game in normal form, a strong Nash equilibrium is an α-core strategy.

SOLUTION 7

1. Suppose not. Then there are consumers i and j such that $x_j^* \succ_i x_i^*$. Since (p^*, x^*) is a Walrasian equilibrium, $p^* \cdot x_j^* > p^* \cdot e_i \geq p^* \cdot x_i^*$, a contradiction to the assumption. □

2. Let $x^* \in W^*(\mathcal{E})$ but $x^* \notin CF(\mathcal{E})$. Then there exist disjoint coalitions S_1, S_2, and $(x_i)_{i \in S_1} \in \prod_{i \in S_1} X_i$ such that

(1) $\sum_{i \in S_1}(x_i - e_i) = \sum_{i \in S_2}(x_i^* - e_i)$,
(2) $x_i \succ_i x_i^*$ for every $i \in S_1$.

Let $(p^*, x^*) \in W(\mathcal{E})$. Then $p^* \cdot x_i > p^* \cdot e_i \geq p^* \cdot x_i^*$ for every $i \in S_1$. Therefore, $0 < \sum_{i \in S_1} p^* \cdot (x_i - e_i) = \sum_{i \in S_2} p^* \cdot (x_i^* - e_i)$, a contradiction because $p^* \cdot x_i^* \leq p^* \cdot e_i$ for every $i \in S_2$ implies $\sum_{i \in S_2} p^* \cdot (x_i^* - e_i) \leq 0$. □

3. Let $S_2 = \emptyset$. Then $\sum_{i \in S_2}(x_i - e_i) = 0$ so that the definition of coalitionally fairness is equilivalent to the core.

4.

(1) $C(\mathcal{E}) = \{((x_1, y_1), (x_2, y_2), (0, 0), (0, 0)) \in R_+^8 : y_1 = x_1, x_1 \in [0, 1], x_1 + x_2 = y_1 + y_2 = 1\}$.
(2) $W(\mathcal{E}) = \{((p_x, p_y), ((x_1, y_1), (x_2, y_2), (0, 0), (0, 0))) \in \Delta \times R_+^8 : x_1 = y_1 = p_x \in [0, 1], x_2 = y_2 = 1 - p_x\}$.
(3) It can be shown that

$$\begin{array}{llll} V(\{i\}) = 0 & \forall\, i, & & \\ V(\{1,2\}) = 1 & V(\{1,3\}) = 1/2 & V(\{1,4\}) = 0, & \\ V(\{2,3\}) = 1/2 & V(\{2,4\}) = 0 & V(\{3,4\}) = 0, & \\ V(\{1,2,3\}) = 1 & V(\{1,2,4\}) = 1 & V(\{1,3,4\}) = 1/2 & V(\{2,3,4\}) = 1/2, \\ V(I) = 1. & & & \end{array}$$

It follows that

$$Sh_1 = \tfrac{5}{12}, \quad Sh_2 = \tfrac{5}{12}, \quad Sh_3 = \tfrac{1}{6}, \quad Sh_4 = 0.$$

At the value allocation, it must hold that

$$\min\{x_1, y_1\} = 5/12,\ \min\{x_2, y_2\} = 5/12,\ (x_3 + y_3)/2 = 1/6.$$

Therefore, the value allocation is given by

$$((x_1^*, y_1^*), (x_2^*, y_2^*), (x_3^*, y_3^*), (x_4^*, y_4^*)) = ((5/12, 5/12), (5/12, 5/12), (1/6, 1/6), (0, 0)).$$

(4) The third consumer and the fourth consumer have the same utility functions and initial endowments but their value allocations are different.

(5) Let $S_1 = \{4\}$, $S_2 = \{3\}$, and $x_4 = (1/6, 1/6)$. Then $x_4 - e_4 = (1/6, 1/6) = x_3^* - e_3$ and $u_4(1/6, 1/6) = 1/6 > 0 = u_4(x_4^*, y_4^*)$.

5. Let x^* be a strong Nash equilibrium but not an α-core strategy. Then there exist a coalition S and $(x_i)_{i\in S} \in \prod_{i\in S} X_i$ such that for every $i \in S$, $u_i((x_i)_{i\in S}, (x_i')_{i\in I\setminus S}) > u_i(x^*)$ for all $(x_i')_{i\in I\setminus S} \in \prod_{i\in I\setminus S} X_i$. The choice of $(x_i')_{i\in I\setminus S} = (x_i^*)_{i\in I\setminus S}$ implies that x^* is not a strong Nash equilibrium, a contradiction. □

HOMEWORK 8

♠ Consider an exchange economy with differential information : $\mathcal{E} = \{(X_i, u_i, e_i, \mathcal{F}_i, \mu) : i \in I\}$, where $X_i = R_+^\ell$ and $\mathcal{F}_i$ is a finite partition of Ω for every i. Define

- $SCC(\mathcal{E})$: the set of strong coarse core allocations.
- $CC(\mathcal{E})$: the set of coarse core allocations.
- $PC(\mathcal{E})$: the set of private core allocations.
- $FC(\mathcal{E})$: the set of fine core allocations.
- $WFC(\mathcal{E})$: the set of weak fine core allocations.

1. Show that $FC(\mathcal{E}) \subset PC(\mathcal{E})$.

2. Show that $PC(\mathcal{E}) \subset CC(\mathcal{E})$.

3. Show that $SCC(\mathcal{E}) \subset CC(\mathcal{E})$.

4. Show that $FC(\mathcal{E}) \subset WFC(\mathcal{E})$.

5. Consider a one-good economy with differential information where there are three agents and three different states of nature that occur with equal probability.

$$\begin{array}{lll} u_1(x) = \log x, & e_1 = (2, 20, 20), & \mathcal{F}_1 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\} \\ u_2(x) = \log x, & e_2 = (22, 22, 1), & \mathcal{F}_2 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\} \\ u_3(x) = \log(x+1), & e_3 = (0, 0, 0), & \mathcal{F}_3 = \{\{\omega_1, \omega_3\}, \{\omega_2\}\} \end{array}$$

(1) Find $CC(\mathcal{E})$.

(2) Find a private core allocation.

(3) Show that in every private core allocation agent 3 gets posititve expected utility.

(4) Find a weak fine core allocation.

(5) Show that in every weak fine core allocation agent 3 does not get anything.

6. Consider a one-good economy with differential information where there are three agents and four states of nature that occur with equal probability.

$$\begin{array}{lll} u_1(x) = \log x, & e_1 = (20, 20, 2, 20), & \mathcal{F}_1 = \{\{\omega_1, \omega_2, \omega_4\}, \{\omega_3\}\} \\ u_2(x) = \log x, & e_2 = (10, 4, 10, 10), & \mathcal{F}_2 = \{\{\omega_1, \omega_3, \omega_4\}, \{\omega_2\}\} \\ u_3(x) = \log(x+1), & e_3 = (0, 0, 0, 0), & \mathcal{F}_3 = \{\{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}\} \end{array}$$

(1) Find $CC(\mathcal{E})$.

(2) Find a private core allocation of $\mathcal{E}$.

(3) Show that in every private core allocation the agent 3 receives positive expected utility.

Now suppose that agent 3 has only trivial information $\mathcal{F}_3 = \{\Omega\}$.

(4) Find $CC(\mathcal{E})$.

(5) Is the allocation of (2) still in $PC(\mathcal{E})$?

7. Consider a one-good economy economy with differential information where there are three agents and three different states of nature that occur with equal probability.

$$
\begin{aligned}
u_1(x) &= \begin{cases} \sqrt{x} & \text{if } \omega = \omega_1, \omega_2, \\ x & \text{if } \omega = \omega_3, \end{cases} && e_1 = (4,4,4), \quad \mathcal{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}, \\
u_2(x) &= \begin{cases} \sqrt{x} & \text{if } \omega = \omega_1, \omega_3, \\ x & \text{if } \omega = \omega_2, \end{cases} && e_2 = (4,4,4), \quad \mathcal{F}_2 = \{\{\omega_1, \omega_3\}, \{\omega_2\}\} \\
u_3(x) &= \ x \quad \text{for every } \omega, && e_3 = (0,0,0), \quad \mathcal{F}_3 = \{\{\omega_1\}, \{\omega_3\}, \{\omega_2\}\}
\end{aligned}
$$

(1) Find $CC(\mathcal{E})$.

(2) Find a private core allocation.

(3) Show that $FC(\mathcal{E}) = \emptyset$.

(4) Find $SCC(\mathcal{E})$.

SOLUTION 8

1. Suppose not. Then there is $x^* \in FC(\mathcal{E})$ with $x^* \notin PC(\mathcal{E})$. Thus there exists a coalition S with $(x_i)_{i\in S} : \Omega \to \prod_{i\in S} X_i$ such that

(1) $x_i - e_i$ is $\mathcal{F}_i$-measurable for every $i \in S$,

(2) $\sum_{i\in S} x_i(\omega) = \sum_{i\in S} e_i(\omega)$, $\mu - a.e.$,

(3) $v_i(x_i) > v_i(x_i^*)$ for every $i \in S$, where $v_i(x_i) = \int_\Omega u_i(\omega, x_i(\omega))d\mu(\omega)$.

Since $x_i - e_i$ is $\mathcal{F}_i$-measurable for every $i \in S$, $x_i - e_i$ is $\bigvee_{i\in S} \mathcal{F}_i$-measurable for every $i \in S$. Hence, S with $(x_i)_{i\in S}$ is a blocking coalition to x^* in the definition of fine core, a contradiction. $\square$

2. Suppose not. Then there is $x^* \in PC(\mathcal{E})$ with $x^* \notin CC(\mathcal{E})$. Thus there exists a coalition S with $(x_i)_{i\in S} : \Omega \to \prod_{i\in S} X_i$ such that

(1) $x_i - e_i$ is $\bigwedge_{i\in S} \mathcal{F}_i$-measurable for every $i \in S$,

(2) $\sum_{i\in S} x_i(\omega) = \sum_{i\in S} e_i(\omega)$, $\mu - a.e.$,

(3) $v_i(x_i) > v_i(x_i^*)$ for every $i \in S$.

Since $x_i - e_i$ is $\bigwedge_{i\in S} \mathcal{F}_i$-measurable for every $i \in S$, $x_i - e_i$ is $\mathcal{F}_i$-measurable for every $i \in S$. Hence, S with $(x_i)_{i\in S}$ is a blocking coalition to x^* in the definition of private core, a contradiction. $\square$

3. Let $x^* \in SCC(\mathcal{E})$. Since x_i^* is $\bigwedge_{i\in I} \mathcal{F}_i$-measurable for every $i \in I$, x_i^* is $\mathcal{F}_i$-measurable for every $i \in I$. Moreover, x^* satisfies the feasibility condition and the no-blocking condition which is the same as that of $CC(\mathcal{E})$. Hence $x^* \in CC(\mathcal{E})$. $\square$

4. Let $x^* \in FC(\mathcal{E})$. Since x_i^* is $\mathcal{F}_i$-measurable for every $i \in I$, x_i^* is $\bigvee_{i\in I} \mathcal{F}_i$-measurable for every $i \in I$. Moreover, x^* satisfies the feasibility condition and the no-coalition condition which are the same as that of $WFC(\mathcal{E})$. Hence $x^* \in WFC(\mathcal{E})$. $\square$

5.

(1) A coarse core allocation x^* is a feasible allocation and x_i^* is $\mathcal{F}_i$-measurable for every i. Furthermore, since each singleton coalition $S = \{i\}$ cannot block x^*, x^* is individually rational. Now consider the coalition $\{1, 2\}$ and (x_1, x_2). Since $\mathcal{F}_1 \wedge \mathcal{F}_2 = \{\Omega\}$, $x_1 - e_1 = (c, c, c)$ and $x_2 - e_2 = (-c, -c, -c)$ with $c \in R$. For this coalition to block x^*, it must be the case that $v_1(x_1) > v_1(x_1^*) \geq v_1(e_1)$. This implies $c > 0$, which, in turn, implies $v_2(x_2) < v_2(e_2)$. Hence $\{1, 2\}$

cannot block x^*. Similarly, $\{1,2\}, \{1,3\}, \{1,2,3\}$ cannot block x^*. (Note that $\mathcal{F}_1 \wedge \mathcal{F}_2 = \mathcal{F}_1 \wedge \mathcal{F}_3 = \bigwedge_{i=1}^{3} \mathcal{F}_i = \{\Omega\}$). Therefore we have

$$CC(\mathcal{E}) = \{x \in L : x \text{ is feasible and individually rational }\}$$

where $L = \prod_{i=1}^{3} L_i$ with $L_i = \{x_i : \Omega \to R_+^3 : x_i \text{ is } \mathcal{F}_i\text{-measurable }\}$ for every i.

(2) An allocation of the following form satisfies the measurability and feasibility condition of $PC(\mathcal{E})$:

$$\begin{aligned} x_1 &= (2+\delta, 20-\varepsilon, 20-\varepsilon), \\ x_2 &= (22-\delta, 22-\delta, 1+\varepsilon), \\ x_3 &= (\quad 0, \quad \varepsilon+\delta, \quad 0), \end{aligned}$$

where $\varepsilon, \delta \geq 0$. Note that $\varepsilon, \delta \geq 0$ is necessary to make everybody better off his endowment due to concave utility functions. To make the allocation survive the grand coalition $\{1,2,3\}$, consider a Pareto problem given by

$$\max_{\substack{\varepsilon \in [0,20] \\ \delta \in [0,22]}} \frac{\lambda}{3}[\log(2+\delta) + 2\log(20-\varepsilon)] + \frac{1-\lambda}{3}[2\log(22-\delta) + \log(1+\varepsilon)]$$

where the third agent's weight is kept zero. When $\lambda = 1/2$, we get $\varepsilon = \delta = 6$ so that

$$x_1^* = (8, 14, 14), \quad x_2^* = (16, 16, 7), \quad x_3^* = (0, 12, 0).$$

We claim that x^* is a private core allocation. Since $v_i(x_i^*) \geq v_i(e_i)$ for $i = 1,2,3$, no singleton coalition can block x. Consider $\{1,2\}$ with (x_1', x_2') as a blocking coalition . Since $x_i' - e_i$ is $\mathcal{F}_i$-measurable for $i = 1,2$ and $x_1' - e_1 = -(x_2' - e_2)$ by feasibility, $x_i' - e_i$ is $\mathcal{F}_1 \wedge \mathcal{F}_2$-measurable. By the same argument in (1), this coalition cannot block x^*. Similarly, $\{1,3\}$ and $\{1,2,3\}$ cannot block x. Hence x^* is a private core allocation.

(3) Suppose not. Let x be a private core allocation with $x_3 = (0,0,0)$. Since $x_3 - e_3 = 0$, $x_1 - e_1 = -(x_2 - e_2)$. Since, however, $x_1 - e_1$ is $\mathcal{F}_1$-measurable and $x_2 - e_2$ is $\mathcal{F}_2$-measurable, $x_1 - e_1$ and $x_2 - e_2$ is $\mathcal{F}_1 \wedge \mathcal{F}_2$-measurable. Therefore, $x_1 - e_1 = (c,c,c)$ and $x_2 - e_2 = (-c,-c,-c)$ for some $c \in R$. If $c \neq 0$, agent 1 or agent 2 blocks x. Thus $x = e$. However, the grand coalition with x^* blocks $x = e$, a contradiction. $\square$

(4) Since $\bigvee_{i=1}^{3} \mathcal{F}_i = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$, we have no information restriction on the weak fine core. Note that since agent 1 and agent 2 pool their information to

have the complete information, there is no incentive to give their endowments to get an additional information. With this intutition, consider a Pareto problem of $\{1,2\}$:

$$\max_{x_1,x_2\in R^3_+} \frac{\lambda}{3}(\log x_1(\omega_1)+\log x_1(\omega_2)+\log x_1(\omega_3))+\frac{1-\lambda}{3}(\log x_2(\omega_1)+\log x_2(\omega_2)+\log x_2(\omega_3))$$

$$s.t.$$

$$x_1(\omega_1)+x_2(\omega_1)=24$$

$$x_1(\omega_2)+x_2(\omega_2)=42$$

$$x_1(\omega_3)+x_2(\omega_3)=21$$

Let $\lambda = 1/2$. Then we have $x_1^* = x_2^* = (12, 21, 21/2)$. We claim that the allocation $x^* = (x_1^*, x_2^*, x_3^*)$ with $x_3^* = e_3$ is a weak fine core allocation. Since $v_i(x_i^*) \geq v_i(e_i)$ for $i = 1, 2, 3$, x^* survives singleton coalitions. Since (x_1^*, x_2^*) solves the Pareto problem of $\{1,2\}$, x^* survives the coalition $\{1,2\}$. Consider coalitions $\{1,3\}$ or $\{2,3\}$. To make agent 3 better off, the other agent should give positive amount of good in some state but agent 3 has nothing to give him in return, which implies that he is worse off. Therefore, these coalitions cannot block x^*. Finally, consider the grand coalition. To make agent 3 better off, the other agents should give positive amount of good in some state but agent 3 has nothing to give him in return, so that they end up with smaller total endowment than before. This implies that agent 1 or agent 2 is worse off since (x_1^*, x_2^*) solves the Pareto problem given above. Thus the grand coalition cannot block x^*. Hence x^* is a weak fine core allocation.

(5) Suppose not. Let x be a weak fine coalition with $x_3(\omega) > 0$ for some ω. Consider the coalition $\{1,2\}$ with (x_1', x_2') such that

$$x_1' = x_1 + (1/2)x_3, \quad x_2' = x_2 + (1/2)x_3.$$

Note that x_i' is $\bigvee_{i=1}^3 \mathcal{F}_i$-measurable for $i = 1, 2$ since x_i is $\bigvee_{i=1}^3 \mathcal{F}_i$-measurable for $i = 1, 2, 3$. Since $x_1' + x_2' = \sum_{i=1}^3 x_i = e_1 + e_2$ and $v_i(x_i') > v_i(x_i)$ for $i = 1, 2$, the coalition blocks x, a contradiction. □

6.

(1) Since $\mathcal{F}_1 \wedge \mathcal{F}_2 = \mathcal{F}_1 \wedge \mathcal{F}_3 = \mathcal{F}_2 \wedge \mathcal{F}_3 = \bigwedge_{i=1}^3 \mathcal{F}_i = \{\Omega\}$, the same argument as in Problem 5 (1) applies, so that the coarse core is given by

$$CC(\mathcal{E}) = \{x \in L : x \text{ is feasible and individually rational }\}$$

where $L = \prod_{i=1}^{3} L_i$ with $L_i = \{x_i : \Omega \to R^4_+ : x_i$ is $\mathcal{F}_i$-measurable $\}$ for every i.

(2) By the similar argument as in Problem 5 (2), we can verify that x^* with

$$x_1^* = (18, 18, 3, 18), \;\; x_2^* = (9, 6, 9, 9), \;\; x_3^* = (3, 0, 0, 3)$$

is a private core allocation.

(3) Suppose not. Let x be a private core allocation with $x_3 = (0, 0, 0)$. Since $x_3 - e_3 = 0$, $x_1 - e_1 = -(x_2 - e_2)$. Since, however, $x_1 - e_1$ is $\mathcal{F}_1$-measurable and $x_2 - e_2$ is $\mathcal{F}_2$-measurable, $x_1 - e_1$ and $x_2 - e_2$ is $\mathcal{F}_1 \wedge \mathcal{F}_2$-measurable. Therefore, $x_1 - e_1 = (c, c, c)$ and $x_2 - e_2 = ((-c, -c, -c)$ for some $c \in R$. If $c \neq 0$, agent 1 or agent 2 blocks x. Thus $x = e$. However, the grand coalition with x^* blocks $x = e$, a contradiction. $\square$

(4) By the measurabliltiy and feasibility condition,

$$CC(\mathcal{E}) = \{e\}.$$

(5) Since $PC(\mathcal{E}) \subset CC(\mathcal{E})$, the private core allocation x^* in (2) is no longer a private core allocation. In fact, $PC(\mathcal{E}) = \{e\}$.

7.

(1) A coarse core allocation x^* is a feasible allocation and x_i^* is $\mathcal{F}_i$-measurable for every i. Furthermore, since each singleton coalition $S = \{i\}$ cannot block x^*, x^* is individually rational. Now consider the coalition $\{1, 2\}$ and (x_1, x_2). Since $\mathcal{F}_1 \wedge \mathcal{F}_2 = \{\Omega\}$, $x_1 - e_1 = (c, c, c)$ and $x_2 - e_2 = (-c, -c, -c)$ with $c \in R$. For this coalition to block x^*, it must be the case that $v_1(x_1) > v_1(x_1^*) \geq v_1(e_1)$. This implies $c > 0$, which, in turn, implies $v_2(x_2) < v_2(e_2)$. Hence $\{1, 2\}$ cannot block x^*. Similarly, the grand coalition cannot block x^* since $\bigwedge_{i=1}^{3} \mathcal{F}_i = \{\Omega\}$. Consider coalitions $\{1, 3\}$ or $\{2, 3\}$. To make agent 3 better off, the other agent should give positive amount of good in some state but agent 3 has nothing to give him in return, which implies that he is worse off. Therefore, these coalitions cannot block x^*. Therefore we have

$$CC(\mathcal{E}) = \{x \in L : x \text{ is feasible and individually rational } \}$$

where $L = \prod_{i=1}^{3} L_i$ with $L_i = \{x_i : \Omega \to R^3_+ : x_i$ is $\mathcal{F}_i$-measurable $\}$ for every i.

(2) An allocation of the following form satisfies the measurability and feasibility condition of $PC(\mathcal{E})$:

$$x_1 = (4 - \varepsilon, 4 - \varepsilon, 4 + \delta),$$
$$x_2 = (4 - \delta, 4 + \varepsilon, 4 - \delta),$$
$$x_3 = (\varepsilon + \delta, \ \ 0, \ \ 0),$$

where $\varepsilon, \delta \geq 0$. Note that $\varepsilon, \delta \geq 0$ is necessary to make everybody better off his endowment due to the form of utility functions. To make the allocation survive the grand coalition $\{1,2,3\}$, consider a Pareto problem given by

$$\max_{\substack{\varepsilon \in [0,4] \\ \delta \in [0,4]}} \frac{\lambda}{3}[2\sqrt{4-\varepsilon} + (4+\delta)] + \frac{1-\lambda}{3}[2\sqrt{4-\delta} + (4+\varepsilon)]$$

where the third agent's weight is kept zero. When $\lambda = 1/2$, we get $\varepsilon = \delta = 3$ so that

$$x_1^* = (1,1,7), \ \ x_2^* = (1,7,1), \ \ x_3^* = (6,0,0).$$

We claim that x^* is a private core allocation. Since $v_i(x_i^*) \geq v_i(e_i)$ for $i = 1,2,3$, no singleton coalition can block x. Consider $\{1,2\}$ with (x_1', x_2') as a blocking coalition . Since $x_i' - e_i$ is $\mathcal{F}_i$-measurable for $i = 1,2$ and $x_1' - e_1 = -(x_2' - e_2)$ by feasibility, $x_i' - e_i$ is $\mathcal{F}_1 \wedge \mathcal{F}_2$-measurable. By the same argument in (1), this coalition cannot block x^*. Consider coalitions $\{1,3\}$ or $\{2,3\}$. To make agent 3 better off, the other agent should give positive amount of good in some state but agent 3 has nothing to give him in return, which implies that he is worse off. Therefore, these coalitions cannot block x^*. Hence x^* is a private core allocation.

(3) Suppose not. Let x be a fine core allocation. Suppose $x_3 = (0,0,0)$. By the measurability and feasibility condition, we have $x = e$. But the coalition $\{1,2\}$ with $x_1' = (4,0,8)$ and $x_2' = (4,8,0)$ blocks [25] $x = e$, a contradiction (Note that x_i' is $\mathcal{F}_1 \vee \mathcal{F}_2$-measurable for $i = 1,2$). Now suppose $x_3 > 0$. Then consider a coalition $\{1,2\}$ with $x_i' = x_i + (1/2)x_3$ for $i = 1,2$. Since $v_i(x_i') > v_i(x_i)$ for $i = 1,2$, this coalition blocks x, a contradiction. □

(4) Note that $\bigwedge_{i=1}^3 \mathcal{F}_i = \{\Omega\}$ and e_i is $\bigwedge_{i=1}^3 \mathcal{F}_i$-measurable for every $i \in I$. Then $x_i - e_i = (c_i, c_i, c_i)$ for some $c_i \in R$. Note that $\sum_{i \in I} c_i = 0$. Therefore, if $c_i \neq 0$ for some i, $c_j < 0$ for some j. This implies that x_j is not individually rational (*i.e.* he will block). Therefore, $c_i = 0$ for every $i \in I$. Hence $SCC(\mathcal{E}) = \{e\}$.

[25] We may consider $x_1' = (4, 1/4, 31/4)$ and $x_2' = (4, 31/4, 1/4)$, which is obtained by the Pareto problem of $\{1,2\}$ without informational constraints.