Hedging the Smirk

David S. Bates

University of Iowa and the
National Bureau of Economic Research

October 31, 2005

Associate Professor of Finance
Department of Finance
Henry B. Tippie College of Business
University of Iowa
Iowa City, Iowa 52242-1000

Tel: (319) 353-2288
Fax: (319) 335-3690
email: david-bates@uiowa.edu

Abstract

This article presents a simple "model-free" method for inferring deltas and gammas from implicit volatility patterns. An illustration indicates that Black-Scholes deltas and gammas are substantially biased in the presence of the sort of smirks and smiles evident in stock index options.

Keywords: options, hedging, volatility smirk, volatility skew, delta, gamma

JEL classifications: G13; G11, G12, C65

Post-'87 implicit volatility patterns in stock index options indicate substantial deviation from the Black-Scholes assumption of a lognormal distribution. Out-of-the-money put options trade at high implicit volatilities relative to at-the-money options, which are in turn higher than those from in-the-money puts and out-of-the-money calls. Figure 1 shows the typical volatility "smirk" pattern, using 21-day S&P 500 futures options on June 24, 2005. And while the overall level of volatilities has varied substantially since 1987, with shocks such as the mini-crashes in 1989, 1997 and 1998 having considerable impact, the relatively high pricing of out-of-the-money puts has been a persistent feature of post-'87 options prices. This volatility pattern indicates the market perceives negative skewness in stock returns. Investors are willing to pay substantially for the downside risk protection offered by out-of-the-money put options.

[Figure 1 about here]

Given that the Black-Scholes assumption of constant implicit volatilities across all strike prices is egregiously violated, what are the appropriate deltas and gammas an option market maker should use when hedging option positions? A parametric approach would take alternate negatively skewed distributions, such as the Bates (1991) jump-diffusion model with negative-mean jumps or a stochastic volatility model with negative correlations between price and volatility shocks. Such multiparameter models can be fitted to observed option prices, and the deltas, gammas and other derivatives can be computed given the parameter estimates. A difficulty is that inferring parameters from such models can be computationally expensive – especially for American options with no closed-form solutions.

This article points out that for a broad class of option pricing models, the appropriate deltas and gammas for hedging option positions can be inferred directly from the pattern of implicit volatilities across different strike prices. The key assumption is that the stochastic process of the underlying asset price exhibits constant returns to scale, so that option prices are homogeneous of degree one in the underlying asset price and the strike price. This assumption first appeared in Merton's (1973) derivation of option pricing properties, and is satisfied by most European and American option pricing models. Examples include the Black-Scholes assumption of geometric Brownian motion, Merton's (1976) jump-diffusion process, and most stock and stock index option models with stochastic volatility or stochastic interest rates. The assumption rules out "level illusion:" whether the S&P 500 index is at 500 or 1000 is irrelevant for the distribution of stock market returns.

Conditional upon homogeneity, Euler's theorem indicates that an option's delta can be inferred directly from the option's sensitivity to the strike price:

$$\Delta \equiv \frac{\partial O}{\partial S} = \frac{1}{S}\left(O - XO_X\right), \qquad \textbf{(1)}$$

where O is the option price, S is the underlying asset price or futures price, and $O_X = \partial O / \partial X$. Similarly, the option's gamma can be computed as[1]

[1] Euler's theorem states that a homogeneous function satisfies

$$SO_S + XO_X = O.$$

Rearranging yields the expression for the delta $\Delta = O_S = \partial O / \partial S$. Similarly, since O_S and O_X are homogeneous of degree zero,

$$SO_{SS} + XO_{SX} = 0$$
$$SO_{XS} + XO_{XX} = 0.$$

Eliminating the cross-derivatives $O_{SX} = O_{XS}$ yields the above expression for the gamma $\Gamma = O_{SS}$.

$$\Gamma \equiv \frac{\partial^2 O}{\partial S^2} = \left(\frac{X}{S}\right)^2 O_{XX}. \tag{2}$$

For the Chicago Mercantile Exchange settlement prices for American options on S&P 500 futures, expressions (1) and (2) can be implemented directly to compute appropriate deltas and gammas. Option settlement prices are determined synchronously with each other and with the futures settlement price, so that O_X and O_{XX} can be computed numerically off observed option settlement prices:

$$\begin{aligned} O_X &\approx \frac{O(\bullet,\, X+\Delta X) - O(\bullet,\, X-\Delta X)}{2\Delta X} \\ O_{XX} &\approx \frac{O(\bullet,\, X+\Delta X) - 2O(\bullet,\, X) + O(\bullet,\, X-\Delta X)}{(\Delta X)^2}. \end{aligned} \tag{3}$$

For example, the deltas and gammas associated with July 2005 put options on June 24, 2005 have the values shown in Table 1.

[Table 1 about here]

More generally, one may be trying to assess appropriate deltas and gammas using badly synchronized intradaily or closing price data. Since intradaily option prices fluctuate considerably with the underlying asset price but intradaily implicit volatility patterns are more stable, it is convenient to express option derivatives in terms of the slope and convexity of the volatility function across different strike prices. Define O^{BS} as the European or American option pricing variant of the Black-Scholes model used in computing implicit volatilities $\hat{\sigma}$: $O = O^{BS}(\bullet, \hat{\sigma})$. By the chain rule,

$$O_X = O_X^{BS} + O_{\hat{\sigma}}^{BS} \frac{\partial \hat{\sigma}}{\partial X} \tag{4}$$

$$O_{XX} = O_{XX}^{BS} + 2 O_{X\hat{\sigma}}^{BS} \frac{\partial \hat{\sigma}}{\partial X} + O_{\hat{\sigma}\hat{\sigma}}^{BS} \left(\frac{\partial \hat{\sigma}}{\partial X} \right)^2 + O_{\hat{\sigma}}^{BS} \frac{\partial^2 \hat{\sigma}}{\partial X^2} \tag{5}$$

Plugging these into (1) and (2) above and exploiting the fact that (1) and (2) also hold for the Black-Scholes option prices O^{BS} yields the following expressions for delta and gamma:

$$\Delta = \Delta^{BS} - O_{\hat{\sigma}}^{BS} \frac{X}{S} \frac{\partial \hat{\sigma}}{\partial X} \tag{6}$$

$$\Gamma = \Gamma^{BS} + \left(\frac{X}{S} \right)^2 \left[2 O_{X\hat{\sigma}}^{BS} \frac{\partial \hat{\sigma}}{\partial X} + O_{\hat{\sigma}\hat{\sigma}}^{BS} \left(\frac{\partial \hat{\sigma}}{\partial X} \right)^2 + O_{\hat{\sigma}}^{BS} \frac{\partial^2 \hat{\sigma}}{\partial X^2} \right], \tag{7}$$

where Δ^{BS} and Γ^{BS} are the Black-Scholes delta and gamma computed at that option's implicit volatility.

Evaluating (6) and (7) requires computing the various partial derivatives of the "Black-Scholes" formula, and computing the slope and convexity of the implicit volatility function $\hat{\sigma}(X)$. The former can be done analytically if a European option pricing formula is used when computing implicit volatilities, and numerically if an American option pricing formula is used. It actually does not matter *which* formula is used when computing derivatives, provided that it is the same as used for computing implicit volatilities. Black's (1976) European futures option pricing model is perfectly acceptable even for American options, and has the advantage of providing analytic derivatives with respect to X and $\hat{\sigma}$. Any errors in Δ^{BS} and Γ^{BS} from a failure to take an early-exercise premium into account are corrected by the slope and convexity of the estimated implicit volatility function. In essence, the implicit volatilities are merely serving as monotonic proxies for the associated option prices.[2]

The slope and convexity of the implicit volatility function $\hat{\sigma}(X)$ can be evaluated numerically for the above settlement price data. More typically, however, it will be necessary to estimate a smoothed volatility function from noisy intradaily implicit volatilities. Many methods are viable; the simplest is probably the regression-based approach of Shimko (1993). The implicit volatilities are regressed on the strike price and strike price squared,

$$\hat{\sigma}(X) = A_0 + A_1 X + A_2 X^2 \tag{8}$$

and the desired first and second derivatives can be estimated using the estimated coefficients:

$$\begin{aligned} \frac{\partial \hat{\sigma}}{\partial X} &= \hat{A}_1 + 2\hat{A}_2 X \\ \frac{\partial^2 \hat{\sigma}}{\partial X^2} &= 2\hat{A}_2 . \end{aligned} \tag{9}$$

For example, the regression-based estimate of the volatility function using implicit volatilities from the above settlement data is $\hat{\sigma}(X) = 7.1014 - .0112526\,X + 4.518054 \times 10^{-6} X^2$, and is the line graphed in Figure 1. Alternatively, the volatility function can be estimated in terms of the

[2]The applications in this article use the Barone-Adesi and Whaley (1987) American futures option pricing model, and associated numerical derivatives

moneyness variable $y = X/S$ rather than X, and used in conjunction with appropriately modified versions of (6) and (7).[3] Spines can also be used, to fit the volatility function more exactly.

Expressions (6) and (7) indicate that the Black-Scholes delta and gamma are biased estimates of the true values if there is substantial slope to the volatility function across different strike prices – which is, of course, the case for stock index options. Since the volatility smirk is downward sloping for low strike prices and the Black-Scholes "vega" $O_{\hat{\sigma}}^{BS}$ is positive, call and put deltas computed using Black-Scholes implicit volatilities and hedge ratios *understate* the true deltas of low-strike options. Figure 2 compares the Black-Scholes deltas with those computed using the regression-based estimate of the volatility function, while Figure 3 compares the gammas.

[Figure 2 about here]

[Figure 3 about here]

Two caveats are in order regarding this method of computing deltas and gammas. First, the method is heavily dependent upon the assumption of homogeneity. And although this property is desirable to ensure returns are stationary, and is satisfied by many popular American and European option pricing models, there do exist models that do not possess this property. Examples include the constant elasticity of variance model of Cox and Rubinstein (1985, pp.361-4), and implied binomial trees models such as Dupire (1994), Derman and Kani (1994), and Rubinstein (1994). Alternate methods of computing option derivatives are necessary for such models. However, nonhomoge-

[3]For instance, the appropriate delta becomes $\Delta = \Delta^{BS} - y \dfrac{O_{\hat{\sigma}}^{BS}}{S} \dfrac{\partial \hat{\sigma}}{\partial y}$, where $O_{\hat{\sigma}}^{BS}/S$ depends only on y given the homogeneity of $O_{\hat{\sigma}}^{BS}$ in S and X.

neous models also imply that at-the-money implicit volatilities are nonstationary, contrary to the mean reversion evident in plots of implicit volatilities over time.[4]

Second, while the proposed methodology may be able to infer the deltas and gammas perceived by the market, that does not mean the market is correct. If options are mispriced, it is probable that the implicit deltas and gammas are also erroneous. Identifying mispriced options does of course require an assessment of what are the correct prices – i.e., a proprietary model. Even in this case, however, the proposed method of computing implicit deltas and gammas may serve as an informative diagnostic for comparison with those estimated using a proprietary model.

[4]Nonhomogeneous models imply that the at-the-money option/asset price ratio O^{ATM}/S depends upon the (nonstationary) underlying asset price S. Consequently, the implicit volatility computed from this ratio under such models must also be nonstationary.

References

Barone-Adesi, G., Whaley, R.E., 1987. Efficient analytic approximation of American option values. Journal of Finance 42, pp. 301-20.

Bates, D. S., 1991. The crash of '87: was it expected? The evidence from options markets. Journal of Finance 46, pp. 1009-44.

Black, F., 1976. The pricing of commodity contracts. Journal of Financial Economics 3, pp. 167-79.

Cox, J.C., Rubinstein, M., 1985. Options Markets. Prentice-Hall, Englewood Cliffs, New Jersey.

Derman, E., Kani, I., 1994. Riding on a smile. Risk 7, pp. 32-9.

Dupire, B., 1994. Pricing with a smile. Risk 7, pp. 18-20.

Merton, R.C., 1973. Theory of rational option pricing. Bell Journal of Economics and Management Science 4, pp. 141-83.

Merton, R.C., 1976. Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics 3, pp. 125-44.

Rubinstein, M., 1994. Implied binomial trees. Journal of Finance 49, pp. 771-818.

Shimko, D., 1993. Bounds of probability. Risk 6, pp. 33-7.

Table 1

July '2005 put options on S&P 500 futures
Settlement prices: June 24, 2005

		Implicit parameters		
Strike Price	Put Price	Volatility ($\hat{\sigma}$)	Delta (Δ)	Gamma (Γ)
1125	1.05	15.96%		
1130	1.20	15.48%	-0.032	0.0018
1135	1.40	15.05%	-0.037	0.0000
1140	1.60	14.54%	-0.042	0.0018
1145	1.85	14.05%	-0.051	0.0018
1150	2.15	13.57%	-0.070	0.0056
1155	2.60	13.23%	-0.090	0.0019
1160	3.10	12.83%	-0.104	0.0038
1165	3.70	12.42%	-0.133	0.0076
1170	4.50	12.09%	-0.172	0.0077
1175	5.50	11.81%	-0.202	0.0039
1180	6.60	11.42%	-0.241	0.0117
1185	8.00	11.11%	-0.301	0.0118
1190	9.70	10.83%	-0.360	0.0119
1195	**11.70**	**10.55%**	**-0.430**	**0.0160**
1200	14.10	10.34%	-0.510	0.0161
1205	16.90	10.18%	-0.591	0.0162
1210	20.10	10.09%	-0.661	0.0123
1215	23.60	9.97%	-0.722	0.0124
1220	27.40	9.87%	-0.773	0.0083
1225	31.40	9.67%		

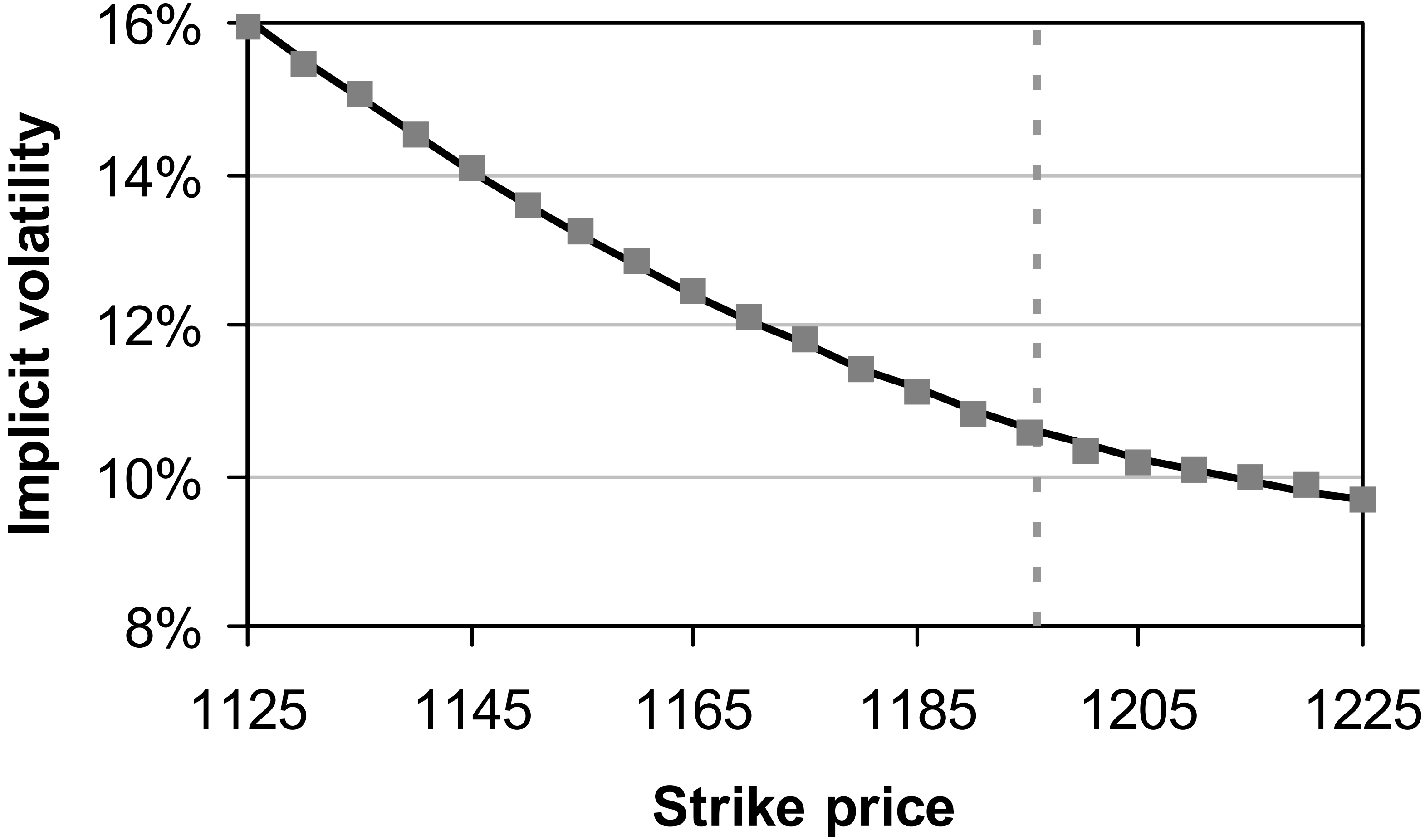
The underlying (Sept. 2005) futures settlement price was 1195.70. Implicit volatilities were computed using the Barone-Adesi and Whaley (1987) American option pricing formula for 21-day options, with a 3.30% interest rate.

Captions for figures

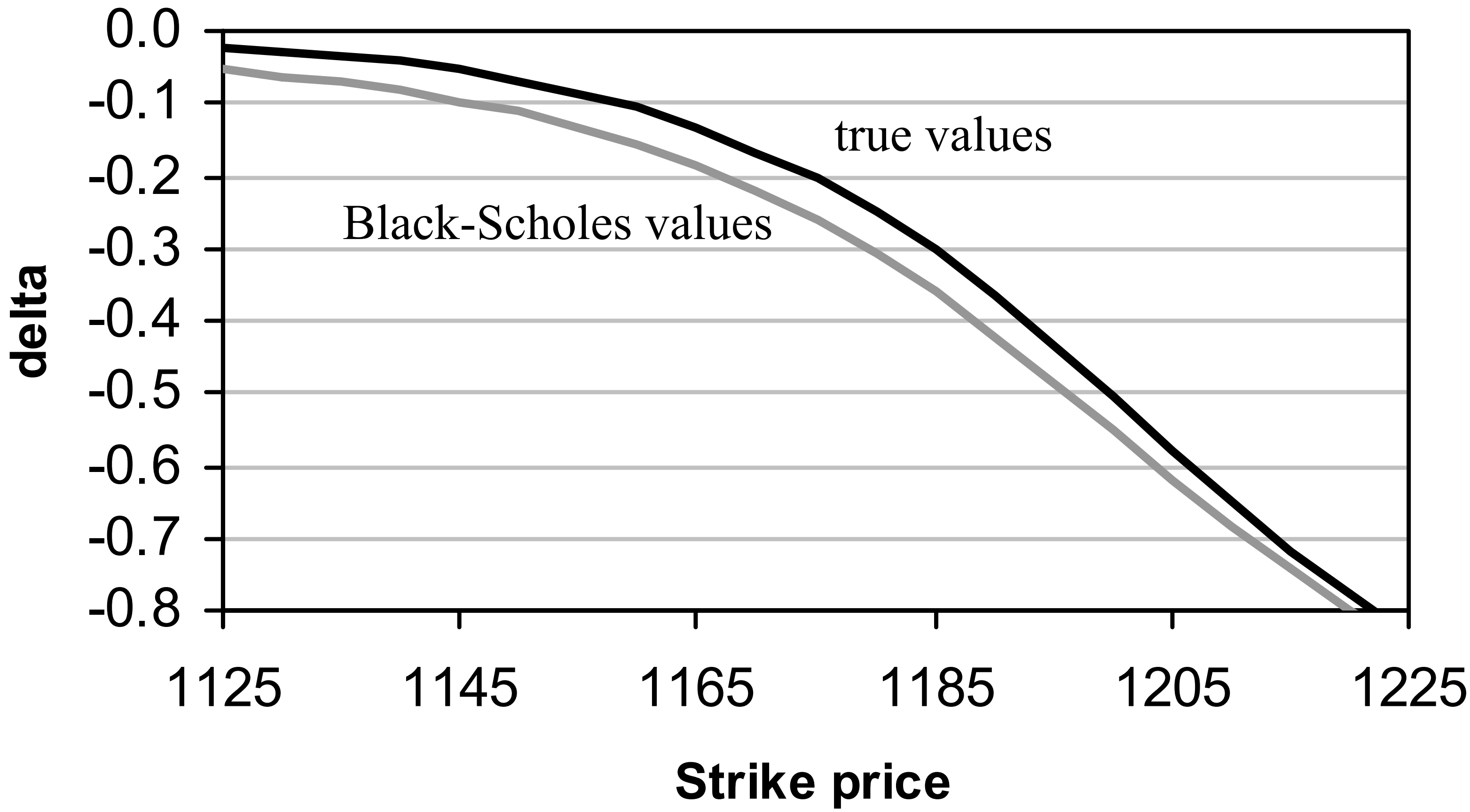
Figure 1. The volatility "smirk." Implicit volatilities from 21-day put options on S&P 500 futures; June 24, 2005.

Figure 2. Implicit deltas. Put options on S&P 500 futures: June 24, 2005.

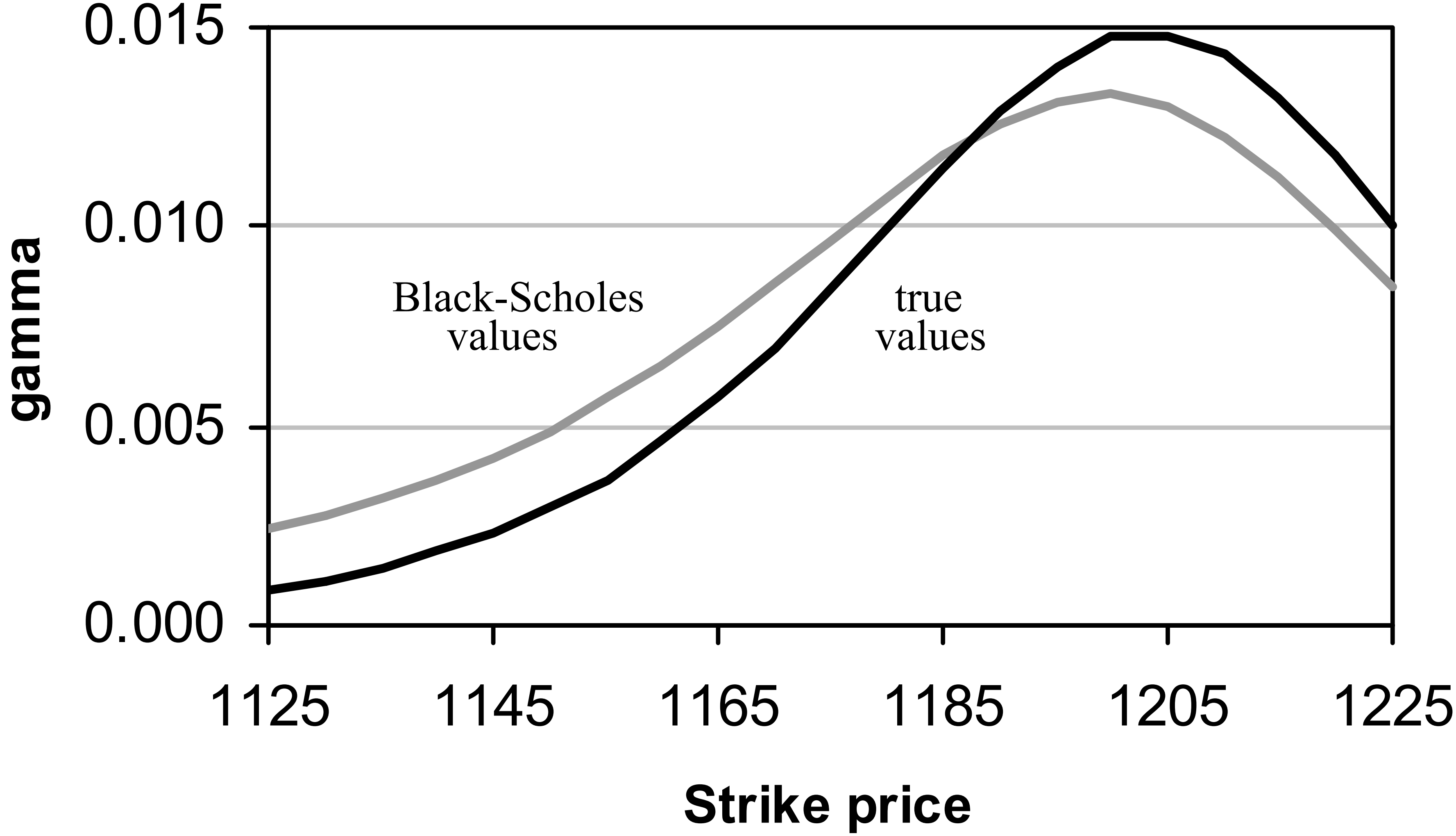
Figure 3. Implicit gammas. Put options on S&P 500 futures: June 24, 2005.



16%
14%
12%
10%
8%
Implicit volatility
1125
1145
1165
1185
1205
1225
Strike price



0.0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
delta
true values
Black-Scholes values
1125
1145
1165
1185
1205
1225
Strike price



0.015
0.010
0.005
0.000
gamma
Black-Scholes values
true values
1125
1145
1165
1185
1205
1225
Strike price