Strengthened(?) Quadratic Programming Bounds for QAP

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The Problem

$$\text{QAP}(A, B, C): \quad \begin{array}{ll} \min & \operatorname{tr}(AXB + C)X^T \\ \text{s.t.} & X \in \Pi, \end{array}$$

where A, B and C are $n \times n$ matrices, tr denotes the trace of a matrix, and Π is the set of $n \times n$ permutation matrices.

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- Main difficulty is obtaining good bounds in reasonable time.

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- Most successful B&B implementations to date have utilized GLB, RLT1/2 and QPB.
- SDP bounds can be very strong at root, but currently too expensive for implementation in B&B.

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$$\begin{aligned} \mathrm{PB} &= \min_{\hat{X} \in \mathcal{O}} \operatorname{tr} \hat{A}\hat{X}\hat{B}\hat{X}^T + (2/n) \min_{X \in \Pi} (AEB) \bullet X \; - \; \gamma \\ &= \langle \hat{\lambda}, \hat{\sigma} \rangle_- \; + \; (2/n) \langle Ae, Be \rangle_- \; - \; \gamma, \end{aligned}$$

where $\langle x, y \rangle_- = \min_\pi \Sigma_{i=1}^n x_i y_{\pi(i)}$, and $\gamma = (e^T Ae)(e^T Be)/n^2$.

Theorem (A-W 2000)

$$\begin{aligned}\langle \hat{\lambda}, \hat{\sigma} \rangle_- = \ & \max \ \operatorname{tr} \hat{S} + \operatorname{tr} \hat{T} \\ & \text{s.t.} \ \ (\hat{B} \otimes \hat{A}) - (I \otimes \hat{S}) - (\hat{T} \otimes I) \succeq 0.\end{aligned}$$

Moreover for any feasible $\hat{S}$, $\hat{T}$ and orthonormal $\hat{X}$,

$$\operatorname{tr} \hat{A}\hat{X}\hat{B}\hat{X}^T = \operatorname{tr} \hat{S} + \operatorname{tr} \hat{T} + \mathbf{vec}(\hat{X})^T \hat{Q}\, \mathbf{vec}(\hat{X}),$$

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To compute QPB, use *optimal* $\hat{S}$, $\hat{T}$ to define $\hat{Q}$, then get

$$\text{QPB} \ = \ \langle \hat{\lambda}, \hat{\sigma} \rangle_- + z(\hat{Q}) - \gamma$$

$$\begin{array}{rcll} z(\hat{Q}) & = & \min & \mathbf{vec}(\hat{X})^T \hat{Q}\, \mathbf{vec}(\hat{X}) + (2/n)(AEB) \bullet X \\ & & \text{s.t.} & \hat{X} = V^T X V \\ & & & Xe = X^T e = e, \ \ X \geq 0. \end{array}$$

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- Dual information associated with solution of QP very helpful for branching decisions.
- QPB successfully used in first solution of nug30 and several other previously unsolved problems.
- Main problem: would be desirable to strengthen bound near root where more computation is practical.

Improving QPB

To improve QPB could consider "outer" maximization problem that varies $\hat{S}$, $\hat{T}$.

$$\begin{aligned} \mathrm{QPB}^{+} = \; & \max \; \operatorname{tr}\hat{S} + \operatorname{tr}\hat{T} - \gamma + z(\hat{Q}) \\ & \text{s.t.} \;\; \hat{Q} = (\hat{B} \otimes \hat{A}) - (I \otimes \hat{S}) - (\hat{T} \otimes I) \succeq 0. \end{aligned}$$

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$$\begin{array}{rl} \text{QPB}^+ = \max & e^T\hat{s} + e^T\hat{t} - \gamma + z(\hat{Q}) \\ \text{s.t.} & \hat{Q} = (U \otimes W) \operatorname{Diag}(\hat{q})(U^T \otimes W^T) \\ & \hat{q} = (\hat{\lambda} \otimes \hat{\sigma}) - (e \otimes \hat{s}) - (\hat{t} \otimes e) \geq 0. \end{array}$$

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- Would be nice to prove that this is always the case!

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Would be more convenient to work with original A, B as opposed to projected $\hat{A}$, $\hat{B}$. However original eigenvalue bound EVB based on A, B is very poor.

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Well known that certain perturbations of data A, B, C preserve value of QAP. For example, consider

$$\begin{aligned} A' &= A + eg^T + ge^T \\ B' &= B + eh^T + he^T \\ C' &= C - 2(Aeh^T + ge^T B + ngh^T + (e^T g)eh^T). \end{aligned}$$

Then $\mathrm{QAP}(A, B, C) = \mathrm{QAP}(A', B', C')$.

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Suggests defining QPB^+ using perturbed data A', B', C' in an effort to further increase bound ...