Comparing Convex Relaxations for Quadratically Constrained Quadratic Programming

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The QCQP problem

Consider a quadratically constrained quadratic program:

$$\begin{array}{lrl} \text{(QCQP)} \quad z^* = & \min & f_0(x) \\ & \text{s.t.} & f_i(x) \le d_i, \quad i = 1, \ldots, q \\ & & x \ge 0, \quad Ax \le b, \end{array}$$

where $f_i(x) = x^T Q_i x + c_i^T x$, $i = 0, 1, \ldots, q$, each Q_i is an $n \times n$ symmetric matrix, and A is an $m \times n$ matrix.

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If $Q_i \succeq 0$ for each i, QCQP is a convex programming problem that can be solved in polynomial time, but in general the problem is NP-Hard. QCQP is a fundamental global optimization problem.

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$$\hat{f}_i(x) = \max\{v^T x \,:\, v^T \hat{x} \le f(\hat{x}) \; \forall \hat{x} \in \mathcal{F}\}.$$

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Let $\widehat{\text{QCQP}}$ be the problem where $\hat{f}_i(\cdot)$ replaces $f_i(\cdot)$, $i = 0, \ldots, q$, and let $\hat{z}$ be the solution value in $\widehat{\text{QCQP}}$.

Second approach to convexifying QCQP is based on linearizing the problem by adding additional variables. Let X denote a symmetric $n \times n$ matrix. Then QCQP can be written

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\begin{aligned}
\text{(QCQP)} \quad z^* = \min\ & Q_0 \bullet X + c_0^T x \\
\text{s.t.}\ & Q_i \bullet X + c_i^T x \le d_i, \quad i = 1, \ldots, q \\
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A convexification of QCQP can then be given in terms of the set

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\mathcal{C} = \mathrm{Co}\left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \,:\, x \in \mathcal{F} \right\}.
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Let $\overline{\text{QCQP}}$ denote problem where $X = xx^T$ is replaced by

$$
Y(x, X) := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}.
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Have two convexifications:

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(\mathrm{Q}\widehat{\mathrm{CQ}}\mathrm{P}) \quad \begin{aligned} \hat{z} = \ \min \ & \hat{f}_0(x) \\ \text{s.t.} \ & \hat{f}_i(x) \le d_i, \quad i = 1, \dots, q \\ & x \in \mathcal{F}. \end{aligned}
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(\overline{\mathrm{QCQP}}) \quad \begin{aligned} \bar{z} = \ \min \ & Q_0 \bullet X + c_0^T x \\ \text{s.t.} \ & Q_i \bullet X + c_i^T x \le d_i, \quad i = 1, \dots, q \\ & Y(x, X) \in \mathcal{C}. \end{aligned}
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Claim: $\hat{z} \le \bar{z}$.

To prove the claim, must relate the different convexifications used to construct $\widehat{\text{QCQP}}$ and $\overline{\text{QCQP}}$.

Theorem 1. *For* $x \in \mathcal{F}$, *let* $f(x) = x^T Qx + c^T x$, *and let* $\hat{f}(\cdot)$ *be the convex lower envelope of* $f(\cdot)$ *on* $\mathcal{F}$. *Then* $\hat{f}(x) = \min\{Q \bullet X + c^T x \,:\, Y(x, X) \in \mathcal{C}\}$.

Theorem 1. *For $x \in \mathcal{F}$, let $f(x) = x^T Qx + c^T x$, and let $\hat{f}(\cdot)$ be the convex lower envelope of $f(\cdot)$ on $\mathcal{F}$. Then $\hat{f}(x) = \min\{Q \bullet X + c^T x : Y(x, X) \in \mathcal{C}\}$.*

Claimed relationship between $\overline{\text{QCQP}}$ and $\widehat{\text{QCQP}}$ is an immediate consequence of Theorem 1. In particular, using Theorem 1, $\widehat{\text{QCQP}}$ could be rewritten in the form

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Corollary 1. *Let $\hat{z}$ and $\bar{z}$ denote solution values in convex relaxations $\widehat{\text{QCQP}}$ and $\overline{\text{QCQP}}$, respectively. Then $\hat{z} \le \bar{z}$.*

Distinction between $\overline{\text{QCQP}}$ and $\widehat{\text{QCQP}}$ is already sharp for $m = n = q = 1$. Consider

$$\begin{aligned} \min\ & x_1^2 \\ \text{s.t.}\ & x_1^2 \geq \tfrac{1}{2} \\ & 0 \leq x_1 \leq 1. \end{aligned}$$

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Written in form of QCQP, $x_1^2 \geq \frac{1}{2}$ is $-x_1^2 \leq -\frac{1}{2}$, and convex lower envelope on $[0, 1]$ is $-x_1$. Relaxation $\widehat{\text{QCQP}}$ is then

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Solution value is $\hat{z} = \frac{1}{4}$. The solution value for $\overline{\text{QCQP}}$ is $\bar{z} = z^* = \frac{1}{2}$. For $x_1 = \frac{1}{2}$, $Y(x_1, x_{11}) \in \mathcal{C}$ for $x_{11} \in [\frac{1}{4}, \frac{1}{2}]$. The solution of $\widehat{\text{QCQP}}$ corresponds to using $x_1 = \frac{1}{2}$ along with $x_{11} = \frac{1}{4}$ for the objective, and $x_{11} = \frac{1}{2}$ for the single nonlinear constraint.

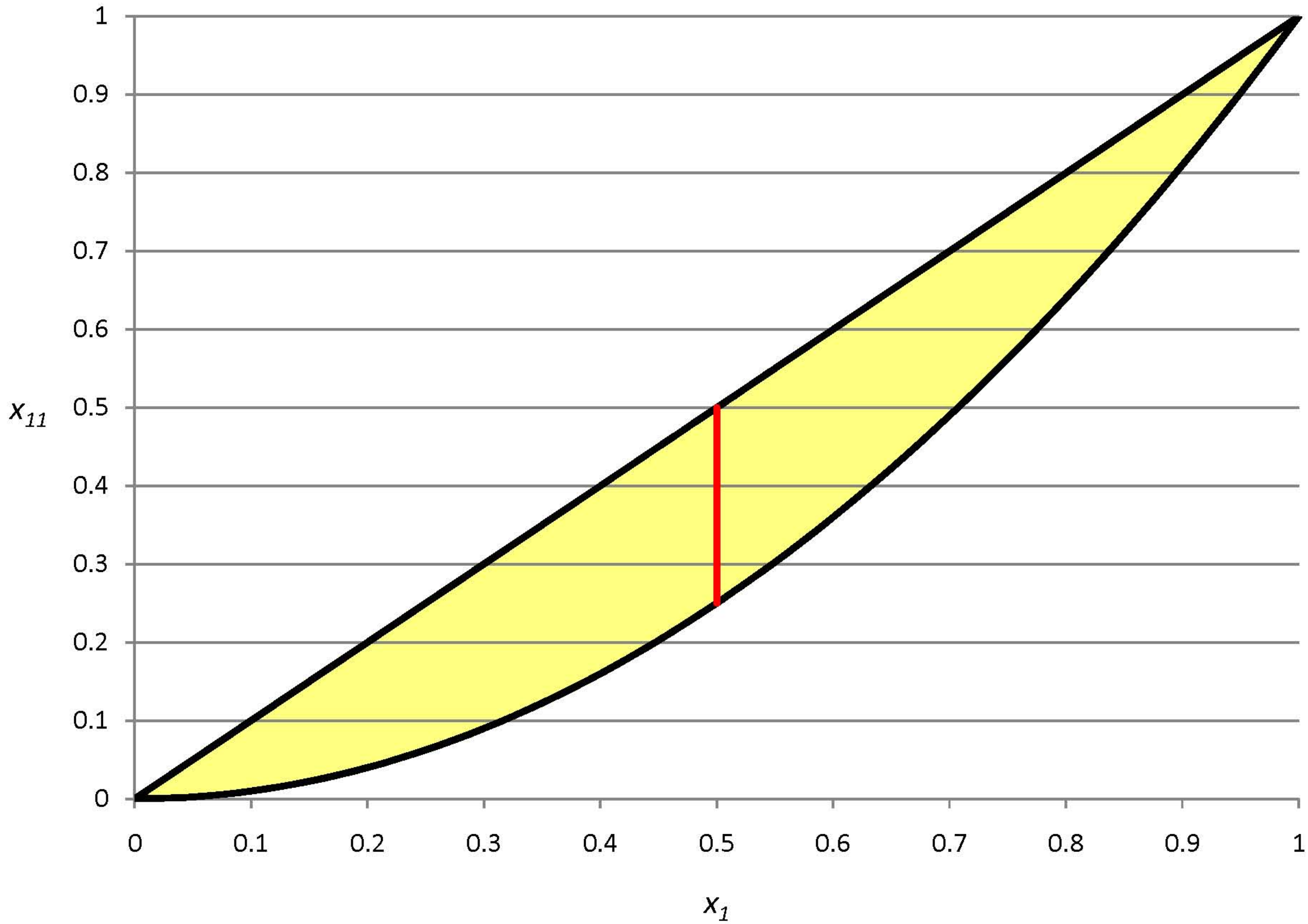

Figure 1: Set $\mathcal{C}$ for example

Two computable relaxations

For a quadratic function $f(x) = x^T Qx + c^T x$ defined on $\mathcal{F} = \{x : 0 \le x \le e\}$, the well-known αBB underestimator is

$$f_\alpha(x) = x^T(Q + \mathrm{Diag}(\alpha))x + (c - \alpha)^T x,$$

where $\alpha \in \Re^n_+$ has $Q + \mathrm{Diag}(\alpha) \succeq 0$. Since $f_\alpha(\cdot)$ is convex, $f_\alpha(x) \le \hat{f}(x)$, $0 \le x \le e$.

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A further relaxation of $\widehat{\mathrm{QCQP}}$ is then given by:

$$\begin{array}{lrl} (\mathrm{QCQP}_{\alpha\mathrm{BB}}) & z_{\alpha\mathrm{BB}} = \min & x^T(Q_0 + \mathrm{Diag}(\alpha_0))x + (c_0 - \alpha_0)^T x \\ & \text{s.t.} & x^T(Q_i + \mathrm{Diag}(\alpha_i))x + (c_i - \alpha_i)^T x \le d_i, \\ & & i = 1, \dots, q \\ & & 0 \le x \le e, \end{array}$$

where each α_i is chosen so that $Q_i + \mathrm{Diag}(\alpha_i) \succeq 0$.

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3. Constraints on the off-diagonal components of $Y(x, X)$ coming from the Boolean Quadric Polytope (BQP), for example, the triangle inequalities for $i \neq j \neq k$,

$$\begin{aligned} x_i + x_j + x_k &\leq x_{ij} + x_{ik} + x_{jk} + 1, \\ x_{ij} + x_{ik} &\leq x_i + x_{jk}, \\ x_{ij} + x_{jk} &\leq x_j + x_{ik}, \\ x_{ik} + x_{jk} &\leq x_k + x_{ij}. \end{aligned}$$

Consider relaxation that imposes $Y(x, X) \succeq 0$ together with the diagonal RLT constraints $\operatorname{diag}(X) \leq x$. Note that these conditions together imply the original bound constraints $0 \leq x \leq e$. Result is:

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Goal is to relate $\mathrm{QCQP}_{\alpha\mathrm{BB}}$ and $\mathrm{QCQP}_{\mathrm{SDP}}$. The following theorem shows that there is a simple relationship between the convexifications used to construct these problems.

Theorem 2. *For* $0 \le x \le e$, *let* $f_\alpha(x) = x^T(Q + \mathrm{Diag}(\alpha))x + (c - \alpha)^T x$, *where* $\alpha \ge 0$ *and* $Q + \mathrm{Diag}(\alpha) \succeq 0$. *Assume that* $Y(x, X) \succeq 0$, $\mathrm{diag}(X) \le x$. *Then* $f_\alpha(x) \le Q \bullet X + c^T x$.

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Immediate corollary proves relationship between $\mathrm{QCQP}_{\alpha\mathrm{BB}}$ and $\mathrm{QCQP}_{\mathrm{SDP}}$ first conjectured by Jeff Linderoth.

Theorem 2. *For* $0 \le x \le e$*, let* $f_\alpha(x) = x^T(Q + \mathrm{Diag}(\alpha))x + (c - \alpha)^T x$*, where* $\alpha \ge 0$ *and* $Q + \mathrm{Diag}(\alpha) \succeq 0$*. Assume that* $Y(x, X) \succeq 0$, $\mathrm{diag}(X) \le x$*. Then* $f_\alpha(x) \le Q \bullet X + c^T x$*.*

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More general convexifications

Consider quadratic function $f(x) = x^T Qx + c^T x$, and $v_j \in \Re^n$, $j = 1, \ldots, k$. Assume for $x \in \mathcal{F}$, $l_j \le v_j^T x \le u_j$. Follows that $(v_j^T x - l_j)(v_j^T x - u_j) \le 0$, or $(v_j^T x)^2 - (l_j + u_j)v_j^T x + l_j u_j \le 0$.

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$$\begin{aligned} Q(\alpha) &= Q + \sum_{j=1}^{k} \alpha_j v_j v_j^T, \\ c(\alpha) &= c - \sum_{j=1}^{k} \alpha_j (l_j + u_j) v_j, \\ p(\alpha) &= \sum_{j=1}^{k} \alpha_j l_j u_j, \end{aligned}$$

and let $f_\alpha(x) = x^T Q(\alpha)x + c(\alpha)^T x + p(\alpha)$.

If $Q(\alpha) \succeq 0$, $f_\alpha(\cdot)$ is a convex underestimator for $f(\cdot)$ on $\mathcal{F}$. Zheng, Sun and Li (2009) refer to functions of the form $f_\alpha(\cdot)$ as DC underestimators, and apply them to convexify the objective in QCQP problems with only linear constraints ($q = 0$).

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Note that the αBB underestimator on $0 \le x \le e$ corresponds to $v_j = e_j$, $l_j = 0$, $u_j = 1$, $j = 1, \ldots, n$. Additional possiblities for v_j include:

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- eigenvectors corresponding to negative eigenvalues of Q,
- transposed rows of the constraint matrix A.

Using underestimators of the form $f_\alpha(\cdot)$, obtain convex relaxation

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(\mathrm{QCQP}_{\mathrm{DC}}) \quad z_{\mathrm{DC}} = \min \ & x^T Q_0(\alpha_0) x + c_0(\alpha_0)^T x + p(\alpha_0) \\
\text{s.t.} \ & x^T Q_i(\alpha_i) x + c_i(\alpha_i)^T x + p(\alpha_i) \le d_i, \\
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We will compare $\mathrm{QCQP}_{\mathrm{DC}}$ to a relaxation of QCQP that combines the semidefiniteness condition $Y(x, X) \succeq 0$ with the RLT constraints on (x, X) that can be obtained from the original linear constraints $x \ge 0$, $Ax \le b$. Refer to this as the "SDP+RLT" relaxation.

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Theorem 3. *Let z_{DC} and $z_{\mathrm{SDP+RLT}}$ denote the solution values in the convex relaxations $\mathrm{QCQP}_{\mathrm{DC}}$ and $\mathrm{QCQP}_{\mathrm{SDP+RLT}}$, respectively. Then $z_{\mathrm{DC}} \le z_{\mathrm{SDP+RLT}}$.*

Conclusion

Approach of approximating $\mathcal{C}$ has substantial advantages over common methodology of approximating convex lower envelopes, *if* solver can handle the constraint $Y(x,X) \succeq 0$.