A note on “5 × 5 Completely positive matrices”

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Abstract

In their paper “5×5 Completely positive matrices,” Berman and Xu [BX04] attempt to characterize which 5×5 doubly nonnegative matrices are also completely positive. Most of the analysis in [BX04] concerns a doubly nonnegative matrix A that has at least one off-diagonal zero component. To handle the case where A is componentwise strictly positive, Berman and Xu utilize an “edge-deletion” transformation of A that results in a matrix $\widetilde{A}$ having an off-diagonal zero. Berman and Xu claim that A is completely positive if and only if there is such an edge-deleted matrix $\widetilde{A}$ that is also completely positive. We show that this claim is false. We also show that two conjectures made in [BX04] regarding 5×5 completely positive matrices are both false.

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1 Introduction

A real symmetric $n \times n$ matrix A is *completely positive* if there exists an entrywise nonnegative $n \times r$ matrix B such that $A = BB^T$. We denote $\mathcal{CP}_n$ as the cone of $n \times n$ completely positive matrices. A real symmetric matrix A is *doubly nonnegative* if A is elementwise nonnegative and positive semidefinite. We denote $\mathcal{DNN}_n$ as the cone of $n \times n$ doubly nonnegative matrices. Obviously we have $\mathcal{CP}_n \subseteq \mathcal{DNN}_n \subseteq \mathcal{DNN}_n^* \subseteq \mathcal{COP}_n$, where $\mathcal{DNN}_n^*$ and $\mathcal{COP}_n$ are dual cones of $\mathcal{DNN}_n$ and $\mathcal{CP}_n$. Matrices in $\mathcal{COP}_n$ are called *copositive.* It is well known that the first and third inclusions are strict if and only if $n \geq 5$ [BSM03]. To understand the difference between $\mathcal{CP}_n$ and $\mathcal{DNN}_n$ it is therefore natural to consider the case of $n = 5$, which has received particular attention in the literature [BX04, Xu01, BAD09].

In [BX04], the authors studied the problem of determining if a given matrix $A \in \mathcal{DNN}_5$ is also in $\mathcal{CP}_5$. If A has a diagonal zero then it is immediate that $A \in \mathcal{CP}_5$, so the diagonal

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components of A may be assumed to be strictly positive. If A has an off-diagonal zero, then after a diagonal scaling and symmetric permutation, A may be assumed to have the form

$$A = \begin{pmatrix} A_{11} & \alpha_1 & \alpha_2 \\ \alpha_1^T & 1 & 0 \\ \alpha_2^T & 0 & 1 \end{pmatrix}, \tag{1}$$

where $A_{11} \in \mathcal{DNN}_3$. The focus of [BX04] is to develop explicit conditions on a matrix A of the form (1) that ensure that $A \in \mathcal{CP}_5$. Many of the conditions developed in [BX04] involve the Schur complement $C = A - \alpha_1\alpha_1^T - \alpha_2\alpha_2^T$. For example, Berman and Xu prove that if $\mu(C)$ is the number of negative entries above the diagonal of C, then $\mu(C) \neq 2 \implies A \in \mathcal{CP}_5$.

To handle the case where $A > 0$ (that is, $a_{ij} > 0$ for all i, j), Berman and Xu introduce the *edge-deletion* operation described in the following definition. For a symmetric $n \times n$ matrix A, let $G(A)$ denote the graph on vertices $\{1, 2, \ldots, n\}$ with edges $\{\{i \neq j\} : a_{ij} \neq 0\}$. Let e_i denote an elementary vector of appropriate dimension whose ith component is equal to one, and $E_{ij} = e_i e_j^T$.

Definition 1. *A matrix $\widetilde{A}$ is an edge-deleted matrix of A if $\widetilde{A} = SAS^T$, where $S = I - \nu E_{ij}$ for some $i \neq j$ and $\nu > 0$, and $G(\widetilde{A})$ is a subgraph of $G(A)$ obtained by deleting at least one of its edges.*

Berman and Xu then claim the following:

Claim 1. [BX04, Theorem 6.1] *Let $A > 0$, $A \in \mathcal{DNN}_5$. Then $A \in \mathcal{CP}_5$ if and only if there exists an edge-deleted matrix of A, $\widetilde{A}$, with $\widetilde{A} \in \mathcal{CP}_5$.*

Using Claim 1, the results of [BX04] based on a matrix of the form (1) could also be applied to a matrix $A > 0$ by first applying the edge-deletion procedure. Unfortunately, in the next section we show via a counterexample that Claim 1 is false. We also describe where the error occurs in the attempted proof of Claim 1 in [BX04]. In section 3 we show that two additional conjectures made in [BX04] regarding matrices in $\mathcal{CP}_5$ of the form (1) are also false.

2 A counterexample to Claim 1

The following 5×5 completely positive matrix appears in [BAD09]. Let

$$N := \frac{\sqrt{2}}{4}\begin{pmatrix} 1&0&0&0&1&1&0&0&1&2 \\ 1&1&0&0&0&2&1&0&0&1 \\ 0&1&1&0&0&1&2&1&0&0 \\ 0&0&1&1&0&0&1&2&1&0 \\ 0&0&0&1&1&0&0&1&2&1 \end{pmatrix}, \quad A := NN^T = \frac{1}{8}\begin{pmatrix} 8&5&1&1&5 \\ 5&8&5&1&1 \\ 1&5&8&5&1 \\ 1&1&5&8&5 \\ 5&1&1&5&8 \end{pmatrix}. \tag{2}$$

Then $A \in \mathcal{CP}_5$, but we will show that there exists no edge-deleted matrix $\widetilde{A}$ of A such that $\widetilde{A} \in \mathcal{CP}_5$. To this end, suppose that $\widetilde{A} = SAS^T$, where $S = I - \nu E_{ij}$, $i \neq j$ and $\nu > 0$. Then

$$\widetilde{A} = A - \nu A e_j e_i^T - \nu e_i e_j^T A + \nu^2 a_{jj} e_i e_i^T,$$

and we immediately obtain

$$\begin{array}{rcll} \widetilde{a}_{kl} & = & a_{kl}, & k \neq i, l \neq i \\ \widetilde{a}_{il} & = & a_{il} - \nu a_{jl}, & l \neq i \\ \widetilde{a}_{ki} & = & a_{ki} - \nu a_{kj}, & k \neq i \\ \widetilde{a}_{ii} & = & a_{ii} - 2\nu a_{ij} + \nu^2 a_{jj}. & \end{array}$$

Note that $\widetilde{A}$ is positive semidefinite by construction, so $\widetilde{a}_{ii} \geq 0$ for any ν. In order to have an off-diagonal zero in $\widetilde{A}$ while maintaining nonnegativity of $\widetilde{A}$, we must therefore have

$$\nu = \min_{l \neq i} \frac{a_{il}}{a_{jl}}. \tag{3}$$

Consider for example $i = 5, j = 3$. Then (3) gives $\nu = \frac{1}{8}$, so $S = I - \frac{1}{8}E_{53}$ and the edge-deleted matrix $\widetilde{A}$ is

$$\widetilde{A} = SAS^T = \frac{1}{8}\begin{pmatrix} 8 & 5 & 1 & 1 & 4.875 \\ 5 & 8 & 5 & 1 & 0.375 \\ 1 & 5 & 8 & 5 & 0 \\ 1 & 1 & 5 & 8 & 4.375 \\ 4.875 & 0.375 & 0 & 4.375 & 7.875 \end{pmatrix}.$$

Clearly $\widetilde{A} \in \mathcal{DNN}_5$, but $\widetilde{A} \notin \mathcal{CP}_5$ because $\widetilde{A} \bullet H := \operatorname{tr} \widetilde{A}H = -\frac{15}{64} < 0$, where $H \in \mathcal{COP}_5$ is the famous Horn matrix given by

$$H := \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}. \tag{4}$$

The matrix H was first proposed by [Hal67] to show that $\mathcal{COP}_5 \setminus \mathcal{DNN}_5^*$ is nonempty. In [BAD09], it was shown that simple transformations of the Horn matrix can be used to separate extreme but not completely positive elements of $\mathcal{DNN}_5$ from $\mathcal{CP}_5$.

The same argument used above for $i = 5, j = 3$ applies to each i, j with $A_{ij} = \frac{1}{8}$; in each case the ratio test (3) gives $\nu = \frac{1}{8}$, and the edge-deleted matrix $\widetilde{A}$ has $H \bullet \widetilde{A} < 0$, demonstrating that $\widetilde{A} \notin \mathcal{CP}_5$.

Next consider $i = 5, j = 1$. Then (3) gives $\nu = \frac{1}{5}$, so $S = I - \frac{1}{5}E_{51}$, and the edge-deleted matrix $\widetilde{A}$ is

$$\widetilde{A} = SAS^T = \frac{1}{8}\begin{pmatrix} 8 & 5 & 1 & 1 & 3.40 \\ 5 & 8 & 5 & 1 & 0 \\ 1 & 5 & 8 & 5 & 0.80 \\ 1 & 1 & 5 & 8 & 4.80 \\ 3.40 & 0 & 0.80 & 4.80 & 6.32 \end{pmatrix}.$$

Once again $\widetilde{A} \in \mathcal{DNN}_5$, but $\widetilde{A} \notin \mathcal{CP}_5$ because $\widetilde{A} \bullet H = -0.06 < 0$. The same argument applies to the other i, j with $A_{ij} = \frac{5}{8}$. We have therefore shown that no edge-deleted matrix of A is in $\mathcal{CP}_5$, as claimed.

Since Claim 1 is false, it is worthwhile to investigate where the error occurs in the attempted proof of [BX04, Theorem 6.1]. The "if" part of Claim 1 is certainly true, and follows easily from the fact that if $S = I - \nu E_{ij}$, where $i \neq j$ and $\nu > 0$, then $S^{-1} = I + \nu E_{ij}$ is nonnegative. To prove the "only if" part of the claim, Berman and Xu use a geometric argument based on interpreting a matrix $A \in \mathcal{CP}_n$ as the Gram matrix of a set of n nonnegative vectors in $\Re^r$, for some r. For $A \in \mathcal{CP}_5$ we then have $a_{ij} = \langle \alpha_i, \alpha_j \rangle$, $1 \leq i, j \leq 5$, where each $\alpha_i \in \Re^r$. The idea of the proof in [BX04] is to construct a new set of vectors $\{\alpha_i'\}_{i=1}^5$ whose Gram matrix corresponds to an edge-deleted matrix of A. This construction requires that unitary rotations be applied to of the vectors $\{\alpha_i\}_{i=1}^5$, but the authors fail to show that these rotations maintain the nonnegativity of $\{\alpha_i'\}_{i=1}^5$ as required to prove that the edge-deleted matrix is in $\mathcal{CP}_5$.

3 Two additional conjectures

In this section we show that two conjectures proposed in [BX04, Section 7] are false. Both conjectures concern a matrix $A \in \mathcal{DNN}_5$ of the form (1). For such a matrix, let C be the Shur complement $C = A_{11} - \alpha_1\alpha_1^T - \alpha_2\alpha_2^T$, and let $\mu(C)$ be the number of negative entries above the diagonal in C. Berman and Xu prove that $\mu(C) \neq 2 \implies A \in \mathcal{CP}_5$, and in [BX04, Section 4] consider the case of $\mu(C) = 2$.

Conjecture 1. *Suppose that $A \in \mathcal{DNN}_5$ has the form (1), with $\mu(C) = \operatorname{rank}(C) = 2$ and $c_{12} > 0$. Then A is completely positive if and only if $\det C[1,2\,|\,1,3] \geq 0$, where*

$$C[1,2\,|\,1,3] = \begin{pmatrix} c_{11} & c_{13} \\ c_{21} & c_{23} \end{pmatrix}.$$

The "if" part is proved to be true in [BX04]. We show the "only if" part is false by a counterexample. Let

$$A := \begin{pmatrix} 2.02 & 1.51 & 0.12 & 0.90 & 0.60 \\ 1.51 & 1.14 & 0.09 & 0.70 & 0.50 \\ 0.12 & 0.09 & 0.57 & 0.40 & 0.10 \\ 0.90 & 0.70 & 0.40 & 1.00 & 0.00 \\ 0.60 & 0.50 & 0.10 & 0.00 & 1.00 \end{pmatrix}, \quad C = \begin{pmatrix} 0.85 & 0.57 & -0.30 \\ 0.58 & 0.40 & -0.24 \\ -0.30 & -0.24 & 0.40 \end{pmatrix}.$$

Clearly $\mu(C) = 2$ and $c_{12} > 0$, and it is easy to verify that $A \in \mathcal{DNN}_5$ and $\operatorname{rank}(C) = 2$. It follows from [BX04, Theorem 2.5] that $A \in \mathcal{CP}_5$. However $\det C[1,2\,|\,1,3] = -0.03$, and therefore Conjecture 1 is false.

In the statement of Conjecture 1 we made the assumption that $c_{12} > 0$, rather than $c_{12} \geq 0$ because $c_{12} > 0$ is assumed throughout [BX04, Section 4]. It is worth noting that the conjecture also fails in the case that $c_{12} = 0$, as shown by the following example. Let

$$A := \begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 4 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

It is easy to show that $A \in \mathcal{DNN}_5$, and therefore $A \in \mathcal{CP}_5$ because $G(A)$ does not contain a 5-cycle. One can easily see that $\mu(C) = \text{rank}(C) = 2$. However, $\det C[1, 2 \,|\, 1, 3] = -1$, and therefore Conjecture 1 is also false in the case that $c_{12} = 0$.

Conjecture 2. *Suppose that $A \in \mathcal{DNN}_5$ has the form* (1) *and is nonsingular. Then $A \in \mathcal{CP}_5$ if and only if it is possible to decrease some of the diagonal entries of A_{11}, resulting in a singular matrix $\widetilde{A}$ with $\widetilde{A} \in \mathcal{CP}_5$.*

The "if" part is shown to be true in [BX04]. To show that the "only if" part is false, consider any matrix $A \in \mathcal{CP}_5$ where $G(A)$ is a 5-cycle[1]. (To construct such a matrix it suffices to take any nonnegative A where $G(A)$ is a 5-cycle and then increase the diagonal components until A is diagonally dominant [BSM03].) It is shown in [BSM03, Chapter 3] that if $A \in \mathcal{CP}_5$ and $G(A)$ is a 5-cycle, then A is nonsingular. Therefore it is impossible to decrease the diagonal entries of A to obtain a singular $\widetilde{A}$ while maintaining $\widetilde{A} \in \mathcal{CP}_5$, and Conjecture 2 is false.

References

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[1] The original draft of the paper considered a particular such matrix. We are grateful to a referee for pointing out the generic nature of this counterexample.